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SUR DE NOUVELLES SOLUTIONS OSCILLATOIRES DES EQUATIONS DE LA GRAVITATION

ALPHONSE MATTE

1. Introduction. Nous savons que dans le champ gravitationnel de la relativité (i.e. défini par les équations $R_{\mu\nu} = 0$) les ondes dites gravitationnelles se propagent avec la vitesse de la lumière. Nous allons maintenant voir, dans le présent travail que si, dans le même continuum, on impose aux potentiels $g_{\mu\nu}$ une oscillation de fréquence assez élevée, il en résulte une onde du type maxwellien. En d'autres termes les éléments de courbure $R_{\alpha\beta\mu\nu}$ vérifient certaines équations, à savoir, $R^{\alpha}_{\beta\sigma\gamma\alpha} = 0$, que nous allons déduire directement de $R_{\mu\nu} = 0$ et qui se réduisent aux équations de Maxwell suivant une approximation qui est précisément basée sur la valeur élevée de la fréquence. Il se trouve alors que l'onde est maxwellienne en première approximation, même si la courbure du milieu de propagation prend une valeur finie, et pour des oscillations dont l'amplitude peut être aussi petite qu'on veut. Nous soulignons ce point parce que dans le cas du résultat classique concernant les ondes gravitationnelles la partie principale de la courbure est précisément due à l'onde et celle-ci se propage dans un milieu qui est presqu'euclidien.

En vue d'obtenir le résultat annoncé il apparaîtra essentiel de séparer l'espace et le temps de sorte que nous adopterons, pour l'élément d'arc, la forme classique:

$$ds^2 = c^2dt^2 - d\sigma^2, \tag{1.1}$$

avec

$$d\sigma^2 = a_{rs}dx^r dx^s \qquad (r, s = 1, 2, 3)$$

dans laquelle les fonctions a_{rs} et c dépendent des quatre variables x^r et t (t pour x^0). Nous verrons alors comment il est possible de remplacer les équations $R_{\mu\nu} = 0$ par un système invariant par rapport aux variables d'espace x^r et dans lequel le temps figure comme un paramètre n'ayant plus le caractère tensoriel. Ce système fournira ensuite de nouvelles équations dans lesquelles les éléments de courbure $R_{\alpha\beta\mu\nu}$ seront groupés en deux tenseurs E_{rs} et H_{rs} tridimensionnels ($r, s = 1, 2, 3$) et de second ordre qui, en première approximation, et du point de vue de la structure des équations, joueront le même rôle que les vecteurs **E** et **H** dans les équations de Maxwell.

2. Notations. Nous rencontrerons des équations invariantes dans l'espace-temps ainsi que d'autres invariantes seulement dans les sections spatiales. Pour faire la distinction nous conviendrons que les lettres grecques employées comme indices doivent prendre les quatre valeurs 0, 1, 2, 3, alors que les caractères latins seront limités aux seules valeurs 1, 2, 3.

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Par ailleurs tout symbole tensoriel surmonté d'une barre comme par exemple:

$$\bar{R}_{\mu\nu},\ \bar{R}_{\alpha\beta\mu\nu},\ \bar{\Gamma}_{\alpha\beta},$$

représentera un système 4-dimensionnel de l'espace-temps, alors que pour les systèmes correspondant des sections spatiales la barre sera omise. De cette façon il sera possible de distinguer entre un système de l'espace 3-dimensionnel tel que R_{ursv} et un autre système 3-dimensionnel comme $\bar{R}_{ursv}$, représentant une partie des composantes du tenseur $\bar{R}_{\alpha\beta\mu\nu}$.

Cependant nous emploierons deux lettres différentes pour représenter les tenseurs métriques: a savoir $g_{\mu\nu}$ pour l'espace temps et a_{rs} pour les sections spatiales. Il est entendu qu'on peut toujours choisir les coordonnées pour que $g_{0i} = 0$ de sorte qu'on a l'élément d 'arc (1.1) dans lequel:

$$g_{00} = c^2, \quad a_{rs} = -g_{rs}. \tag{2.1}$$

Alors le système a_{rs} détermine la métrique de toute une famille de sections spatiales paramétrées suivant le temps t, et l'espace 3-dimensionnel ainsi déterminé sera souvent représenté par le symbole $E(3)$.

Un calcul facile donne:

$$\begin{aligned} g &= -c^2 a, \qquad a = ||a_{rs}||, \\ g^{00} &= c^{-2},\ g^{0i} = 0,\ g^{rs} = -a^{rs}. \end{aligned} \tag{2.1a}$$

Nous écrirons

$$A_{rs} = \frac{1}{2c}\frac{\partial a_{rs}}{\partial t}. \tag{2.2}$$

Le système ainsi défini est un tenseur de $E(3)$ puisque d'une part, les éléments $\partial x^i/\partial x'^k$ d'une transformation ne dépendant pas du temps et l'opérateur $\partial/\partial t$ d'autre part sont permutables. Cependant il est important de noter que la relation correspondante écrite au moyen des composantes contravariantes définirait le tenseur de signe contraire. Effectivement si on dérive les deux membres de l'identité

$$a^{rs} = a_{mn}a^{mr}a^{ns},$$

on obtient

$$\frac{\partial a^{rs}}{\partial t} = a^{mr}a^{ns}\frac{\partial a_{mn}}{\partial t} + \delta_m{}^s\frac{\partial a^{mr}}{\partial t} + \delta_n{}^r\frac{\partial a^{ns}}{\partial t},$$

d'où

$$-\frac{\partial a^{rs}}{\partial t} = a^{mr}a^{ns}\frac{\partial a_{mn}}{\partial t},$$

c'est-à-dire

$$A^{rs} = -\frac{1}{2c}\frac{\partial a^{rs}}{\partial t}. \tag{2.2a}$$

Ce tenseur est évidemment symétrique.

Dans $g_{00} = c^2$, la fonction $c(x^r,t)$ est un invariant de $E(3)$ et son gradient peut se présenter dans nos équations invariants 3-dimensionnelles comme un vecteur. Nous adopterons la notation:

$$\frac{\partial c}{\partial x^\alpha} = c_\alpha, \quad \frac{\partial c}{\partial t} = c_0.$$

Pour ce qui est des symboles de Christoffel nous adopterons la notation:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right), \quad \Gamma_{\alpha\beta}{}^\mu = g^{\mu\gamma}\Gamma_{\alpha\beta\gamma}.$$

Alors avec l'aide de $g_{rs} = -a_{rs}$ dans (2.1) nous avons immédiatement: $\Gamma_{rst} = -\Gamma_{rst}$. Par ailleurs

$$\Gamma_{000} = \frac{1}{2}\frac{\partial g_{00}}{\partial t} = cc_0,$$

et de la même façon il est aisé de vérifier les résultats qui suivent:

$$\begin{aligned}
&\Gamma_{000} = cc_0, \quad \Gamma_{rs0} = \frac{1}{2}\frac{\partial a_{rs}}{\partial t},\\
&\Gamma_{00r} = -cc_r, \quad \Gamma_{r0s} = -\frac{1}{2}\frac{\partial a_{rs}}{\partial t},\\
&\Gamma_{r00} = cc_r, \quad \Gamma_{rst} = -\Gamma_{rst},\\
(2.3)\qquad &\begin{cases}
\Gamma_{00}{}^0 = \dfrac{c_0}{c}, & \Gamma_{rs}{}^0 = \dfrac{1}{c}A_{rs},\\
\Gamma_{00}{}^r = a^{rs}cc_s, & \Gamma_{r0}{}^s = cA_r{}^s,\\
\Gamma_{r0}{}^0 = \dfrac{c_r}{c}, & \Gamma_{rs}{}^t = \Gamma_{rs}{}^t.
\end{cases}
\end{aligned}$$

3. Les équations $\bar{R}_{\mu\nu} = 0$ sous forme invariante dans $E(3)$. Considérons le système $\bar{R}_{uvst}$ qui représente seulement une partie des composantes du tenseur $\bar{R}_{\alpha\beta\mu\nu}$; nous allons voir que nous avons là un tenseur de $E(3)$ aussi longtemps que rien n'est changé dans la séparation de l'espace et du temps. Cela veut dire que nous considérons des transformations de coordonnées du type

$$x'^0 = x^0 = t, \quad \frac{\partial x^0}{\partial x'^0} = 1, \quad \frac{\partial x^0}{\partial x'^r} = \frac{\partial x^r}{\partial x'^0} = 0.$$

Alors la relation

$$\bar{R}'_{uvst} = \bar{R}_{\alpha\beta\mu\nu}\frac{\partial x^\alpha}{\partial x'^u}\frac{\partial x^\beta}{\partial x'^v}\frac{\partial x^\mu}{\partial x'^s}\frac{\partial x^\nu}{\partial x'^t}$$

donne immédiatement (puisqu'aucun terme significatif n'est obtenu lorsque les indices muets prennent la valeur 0)

$$\bar{R}'_{uvst} = \bar{R}_{abmn}\frac{\partial x^a}{\partial x'^u}\frac{\partial x^b}{\partial x'^v}\frac{\partial x^m}{\partial x'^s}\frac{\partial x^n}{\partial x'^t},$$

ce qui prouve notre avancé.

Les autres composantes de $\bar{R}_{\alpha\beta\mu\nu}$ donnent encore deux de ces tenseurs 3-dimensionnels dont l'ordre est égal au nombre des indices qui n'ont pas été remplacés par la valeur 0. La vérification du caractère tensoriel s'effectue comme précédemment et nous avons en premier lieu

$$\bar{R}'_{r0\theta\iota} = \bar{R}_{\alpha\beta\mu\nu} \frac{\partial x^\alpha}{\partial x'^r} \frac{\partial x^\beta}{\partial x'^0} \frac{\partial x^\mu}{\partial x'^\theta} \frac{\partial x^\nu}{\partial x'^\iota},$$

où les sommes donnent des termes non nuls seulement pour $\beta = \mu = 0$ de sorte que

$$\bar{R}'_{r0\theta\iota} = \bar{R}_{\alpha 00\alpha} \frac{\partial x^0}{\partial x'^r} \frac{\partial x^\alpha}{\partial x'^\iota}.$$

Enfin on montre, toujours de la même façon que le système R_{r0mn} est un tenseur de troisième ordre de $E(3)$.

Notons qu'on peut toujours faire usage des propriétés de symétrie de $R_{\alpha\beta\mu\nu}$ qui donneront par example

$$\bar{R}_{r0mn} = -\bar{R}_{0rmn} = \bar{R}_{mnr0}.$$

En conséquence tous les éléments du tenseur $\bar{R}_{\alpha\beta\mu\nu}$ se trouvent ainsi regroupés en trois tenseurs 3-dimensionnels de l'espace $E(3)$. Le passage au composantes contravariantes sera illustré dans des calculs subséquents.

Nous emploierons la même technique pour séparer les dérivées $\bar{R}_{\alpha\beta\mu\nu,\sigma}$ en un certain nombre de tenseurs 3-dimensionnels comme, par exemple, $\bar{R}_{ur\theta\iota,\bar{\imath}}$. Ici le dernier indice est surmonté d'une barre pour spécifier qu'il s'agit de la dérivation covariante par rapport à l'espace-temps. Le lecteur n'aura aucune peine à s'assurer que notre raisonnement sur les transformations de coordonnées dans $E(3)$ vaut encore pour établir le caractère tensoriel des cinq indices. Bien entendu ce nouveau tenseur est distinct de $\bar{R}_{ur\theta\iota,\bar{s}}$ obtenu par la dérivation dans $E(3)$ et nous verrons plus loin comment on peut développer le premier de façon à tout ramener à la dérivation dans ce dernier espace.

Occupons nous d'abord de la séparation des équations $\bar{R}_{\mu\nu} = 0$ tel qu'annoncé. Une partie du système s'écrit

$$\bar{R}_{rs} = g^{\alpha\beta}\bar{R}_{\alpha r s\beta} = 0$$

ou, avec l'aide de (2.1a),

$$-a^{ut}\bar{R}_{urst} + c^{-2}\bar{R}_{0rs0} = 0. \tag{3.1}$$

Pour le reste on a les équations

$$\bar{R}_{0n} = g^{\alpha\beta}\bar{R}_{\alpha 0n\beta} = 0,$$

$$\bar{R}_{00} = g^{\alpha\beta}\bar{R}_{\alpha 00\beta} = 0,$$

qui donnent immédiatement

$$a^{uv}\bar{R}_{u0nv} = 0, \tag{3.2}$$

$$a^{rs}\bar{R}_{r00s} = 0. \tag{3.3}$$

Ces équations représentent une forme invariante dans $E(3)$ du système $\bar{R}_{\mu\nu} = 0$. On pourrait les écrire en fonction des tenseurs a_{rs} et A_{rs} mais cela ne sera pas nécessaire pour atteindre le but que nous poursuivons.

4. Symétrie des nouveaux tenseurs. Nous utiliserons le tenseur

$$\epsilon^{rst} = a^{-\frac{1}{2}}e^{rst}, \quad \epsilon_{rst} = a^{\frac{1}{2}}e_{rst},$$

où $e_{rst} = e^{rst}$ prennent les valeurs suivantes:

0 si deux quelconques des indices *rst* ont la même valeur.

1 si *rst* représente une permutation paire des nombres 1, 2, 3.

-1 si *rst* représente une permutation impaire des mêmes nombres.

Nous tenons à souligner que nous avons là les deux types de coordonnées d'un même tenseur puisque les deux systèmes coincident en coordonnées normales ($a = 1$). Il est donc entendu que l'on peut passer d'un à l'autre indifféremment, lorsque dans nos équations tensorielles certains indices sont élevés ou abaissés.

Nous avons d'ailleurs:

$$\epsilon_{usr}\epsilon^{usv} = 2\delta_r^{\ v} \tag{4.1}$$

où $\delta_r^{\ s}$ représente la matrice unitaire. La vérification est aisée et il en résulte que ce dernier système est un tenseur mixte.

Un autre tenseur mixte qui nous sera très utile s'introduit comme suit:

$$\delta_{uv}^{\ \ rs} = \epsilon_{uvn}\epsilon^{rsn}. \tag{4.2}$$

On trouve aisément pour la valeur des composantes:

$$\begin{array}{rl} 1 & \text{si } u \neq v, \quad r = u, \quad s = v, \\ -1 & \text{si } u \neq v, \quad r = v, \quad s = u, \\ 0 & \text{dans les autres cas.} \end{array}$$

Posons alors:

$$4E^{rs} = \bar{R}_{uvmn}\epsilon^{uvr}\epsilon^{mns}, \tag{4.3}$$

$$2cH_r^{\ s} = \bar{R}_{r0us}\epsilon^{uvs}. \tag{4.4}$$

Puisque $\bar{R}_{uvmn} = \bar{R}_{mnuv}$ nous avons $E^{sr} = E^{rs}$ et, bien entendu, cela entraine la symétrie des composantes covariantes:

$$E_{sr} = E_{rs}. \tag{4.5}$$

Les équations (4.3) peuvent être résolues pour les composantes $\bar{R}_{uvmn}$: multipliant les deux membres par $\epsilon_{rsb}\epsilon_{xry}$ et faisant usage de (4.2) on a

$$4E^{rs}\epsilon_{abr}\epsilon_{xry} = \bar{R}_{uvmn}\delta_{ab}^{\ \ uv}\delta_{xy}^{\ \ mn},$$

et puisque les seules valeurs utiles pour les indices (uv) sont (ab) et (ba) avec les

valeurs correspondantes $\delta_{ab}{}^{ab} = 1$ et $\delta_{ab}{}^{ba} = -1$ on obtient

$$4E^{rs}\epsilon_{abr}\epsilon_{zys} = (\bar{R}_{abmn} - \bar{R}_{bamn})\delta_{zy}{}^{mn}$$

$$= 2\bar{R}_{abmn}\delta_{zy}{}^{mn}.$$

Finalement on a

$$\bar{R}_{abzy} = E^{rs}\epsilon_{abr}\epsilon_{zys}. \tag{4.6}$$

Par ailleurs (3.1) et (4.6) donnent successivement

$$c^{-2}\bar{R}_{0r0} = a^{us}\bar{R}_{usrs} = a^{us}E^{mn}\epsilon_{usm}\epsilon_{srn}$$

$$= E_m{}^n\epsilon^{sfm}a_{tr}\epsilon_{srn} = E_m{}^n\delta_n{}^{fm}a_{tr}$$

$$= E_s{}^n a_{nr} - E_n{}^n a_{sr}$$

$$= E_{rs} - a_{rs}E_n{}^n.$$

Alors par contraction de la dernière égalité et avec l'aide de (3.3) on a

$$c^{-2}a^{rs}\bar{R}_{0r0} = E_r{}^r - 3E_n{}^n = -2E_n{}^n = 0,$$

d'où les résultats suivants:

$$E_n{}^n = 0, \tag{4.7}$$

$$E_{rs} = c^{-2}\bar{R}_{r00s} = a^{us}\bar{R}_{usrs}. \tag{4.8}$$

On peut noter que l'invariant $E_n{}^n$ s'annule par suite des équations de la relativité alors que le tenseur E_{rs} est symétrique identiquement. Nous allons retrouver des résultats identiques dans le cas de H_{rs} sauf que l'ordre est renversé.

De la relation (4.4) on tire successivement

$$2cH^{rs} = a^{rt}\bar{R}_{t0us}\epsilon^{usv},$$

$$2cH^{rs}\epsilon_{rzn} = a^{rt}\bar{R}_{t0us}\delta_{nr}{}^{us} = 2a^{rt}\bar{R}_{t0nr},$$

qui s'annule à cause de (3.2). Mais $H^{rs}\epsilon_{rzn}$ peut s'annuler seulement si

$$H^{sr} = H^{rs}. \tag{4.9}$$

D'un autre coté

$$2cH_r{}^r = \bar{R}_{r0us}\epsilon^{usr} = -\bar{R}_{0rus}\epsilon^{usr},$$

qui s'annule identiquement par suite de l'identité classique

$$\bar{R}_{\alpha\beta\mu\nu} + \bar{R}_{\alpha\mu\nu\beta} + \bar{R}_{\alpha\nu\beta\mu} = 0.$$

On a donc

$$H_r{}^r = 0. \tag{4.10}$$

Enfin une autre relation qui sera utile par la suite, se déduit de (4.4) en multipliant les deux membres par $\epsilon_{\alpha\mu\nu}$.

$$2cH_\tau^\sigma \epsilon_{\sigma\mu\nu} = \bar{R}_{\tau 0\alpha\beta}\delta_{\mu\nu}^{\alpha\beta} = 2\bar{R}_{\tau 0\mu\nu},$$

d'où

$$\bar{R}_{\tau 0\alpha\beta} = cH_\tau^\mu \epsilon_{\mu\alpha\beta}. \tag{4.11}$$

Les deux tenseurs que nous venons d'introduire sont à la base d'un nouveau formalisme qui va nous permettre de mettre en lumière un aspect jusque là ignoré, de la structure des équations de la gravitation. A vrai dire nous nous intéresserons moins à ces équations qu'a certaines de leurs conséquences différentielles avec lesquelles nous allons tout de suite faire connaissance.

5. Un système d'équations fondamentales. Partons de l'identité de Bianchi,

$$\bar{R}_{\alpha\mu\tau\beta,\sigma} + \bar{R}_{\alpha\mu\beta\sigma,\tau} + \bar{R}_{\alpha\mu\sigma\tau,\beta} = 0.$$

Après contraction avec $g^{\alpha\beta}$ les deux premiers termes s'annulent à cause de $\bar{R}_{\mu\tau} = 0$ et il reste

$$g^{\alpha\beta}\bar{R}_{\alpha\mu\sigma\tau,\beta} = 0. \tag{5.1}$$

Occupons nous maintenant de séparer ces équations suivant la technique déjà employée. Une partie du système peut s'écrire $g^{\alpha\beta}\bar{R}_{\alpha\tau\sigma\beta} = 0$ ou

$$-a^{\mu\nu}\bar{R}_{\mu\tau\sigma\bar{\nu}} + c^{-2}\bar{R}_{0\tau\sigma\bar{0}} = 0. \tag{5.2}$$

Et pour la série au complet nous obtenons encore:

$$-a^{\mu\nu}\bar{R}_{\mu\alpha\beta\bar{\nu}} + c^{-2}\bar{R}_{0\alpha\beta\bar{0}} = 0, \tag{5.3}$$

$$a^{\tau\beta}\bar{R}_{\tau 0\alpha\bar{\beta}} = 0, \tag{5.4}$$

$$a^{\tau\beta}\bar{R}_{\tau 0\beta\bar{\alpha}} = 0. \tag{5.5}$$

Disons maintenant que c'est après avoir transcrit le système (5.2) à (5.5) en fonction des tenseurs $E_{\tau\beta}$ et $H_{\tau\beta}$ qu'on y retrouvera la structure des équations de Maxwell. Bien entendu nous avons ici affaire aux dérivées dans l'espace-temps et il nous faut d'abord développer les termes de façon à tout ramener aux dérivées dans $E(3)$. Nous avons donc recours au développement classique de la dérivée covariante et nous avons par exemple

$$\bar{R}_{\mu\tau\sigma\bar{\alpha}} = \frac{\partial \bar{R}_{\mu\tau\sigma}}{\partial x^\alpha} - \Gamma_{\alpha\mu}^{\ \ \rho}\bar{R}_{\rho\tau\sigma} - \Gamma_{\alpha\tau}^{\ \ \rho}\bar{R}_{\mu\rho\sigma} - \Gamma_{\alpha\sigma}^{\ \ \rho}\bar{R}_{\mu\tau\rho} - \Gamma_{\alpha\bar{0}}^{\ \ \rho}\bar{R}_{\mu\tau\sigma}.$$

Comparons cette égalité avec

$$\bar{R}_{\mu\tau\sigma,\alpha} = \frac{\partial \bar{R}_{\mu\tau\sigma}}{\partial x^\alpha} - \Gamma_{\alpha\mu}^{\ \ \rho}\bar{R}_{\rho\tau\sigma} - \Gamma_{\alpha\tau}^{\ \ \rho}\bar{R}_{\mu\rho\sigma} - \Gamma_{\alpha\sigma}^{\ \ \rho}\bar{R}_{\mu\tau\rho} - \Gamma_{\alpha 0}^{\ \ \rho}\bar{R}_{\mu\tau\sigma}.$$

On en tire, avec l'aide de $\Gamma_{rs}^{t} = \Gamma_{sr}^{t}$,

$$\bar{R}_{urss,\bar{n}} = \bar{R}_{urss,n} - \Gamma_{nu}^{0}\bar{R}_{0rss} - \Gamma_{nr}^{0}\bar{R}_{u0ss} - \Gamma_{ns}^{0}\bar{R}_{ur0s} - \Gamma_{ns}^{0}\bar{R}_{urs0}. \tag{5.6}$$

Par ailleurs on a directement

$$\bar{R}_{0rss,\bar{0}} = \frac{\partial \bar{R}_{0rss}}{\partial t} - \Gamma_{00}^{a}\bar{R}_{arss} - \Gamma_{0r}^{a}\bar{R}_{0ass} - \Gamma_{0s}^{a}\bar{R}_{0ras} - \Gamma_{0s}^{a}\bar{R}_{0rsa}. \tag{5.7}$$

Nous suspendons momentanément les calculs en vue d'introduire une méthode d'approximation qui va nous permettre de simplifier en négligeant un grand nombre de termes. Ces considérations feront l'objet de la section suivante.

6. Ondes à haute fréquence. Nous considérerons d'abord le cas d'un onde dont la fréquence prend une valeur n très élevée et c'est sur ce nombre n que notre méthode d'approximation sera basée. Comme la vitesse de propagation de nos ondes va se révéler égale à l'invariant c, nous pouvons d'avance profiter de ce fait et choisir l'unité de temps de telle sorte que les potentiels c^2 et a_{rs} soient tous du même ordre de grandeur. Nous adopterons donc le centimètre comme unité de longueur et nous choisirons comme unité de temps l'intervalle nécessaire à la lumière pour parcourir cette distance, ce qui entraine évidemment que la valeur de la vitesse de la lumière dans le vide est égale à un. Par ailleurs une fréquence de 10^4 signifie: 10^4 ondes au centimètre aussi bien qu'un nombre égal de périodes par unité de temps. Alors, en vue des considérations qui vont suivre il est important de noter que notre choix d'unités entraine le résultat suivant: non seulement les potentiels c^2 et a_{rs} sont du même ordre de grandeur mais encore leur taux de variation par rapport au temps (par suite de l'oscillation) est du même ordre de grandeur que le taux de variation par rapport à la distance. En d'autres termes les dérivées du premier ordre par rapport au temps et celles par rapport aux variables d'espace prennent toutes des valeurs qui sont du même ordre de grandeur (sauf évidemment au voisinage des singularités; nous considérons le cas d'un domaine dans lequel la courbure reste finie). Pour fixer les idées remarquons qu'une fréquence $n = 10^4$ correspond à une onde dans l'infra-rouge, à peu près à la limite du spectre visible. D'un autre coté, et pour éviter toute confusion avec ces données expérimentales concernant la lumière, rappelons encore une fois que nous voulons simplement imposer aux potentiels g_{rs} une oscillation de fréquence suffisament élevée et montrer qu'alors il en résulte une onde caractérisée en première approximation par des équations ayant même structure que celles du système de Maxwell.

Pour commencer, il sera commode de nous restreindre au cas où l'intensité d'oscillation des potentiels est de l'ordre de n^{-1} et nous prendrons $n > 10^4$. Alors dans un domaine de l'espace-temps s'étendant à un nombre raisonnable d'ondes et de périodes les oscillations de nos potentiels peuvent être représentées en première approximation par des fonctions sinusoïdales. C'est-à-dire qu'il suffit de s'en rapporter à la relation

$$\frac{d}{dx}\left(\frac{1}{2\pi n}\sin 2\pi n x\right) = \cos 2\pi n x$$

pour s'assurer que, si l'intensité d'oscillation des potentiels est du même ordre de grandeur que n^{-1}, les dérivées du premier ordre de ces mêmes potentiels prennent des valeurs du même ordre que n^0, c'est-à-dire qu'elles sont des grandeurs "finies" du point de vue de notre échelle suivant les puissances de n. Par ailleurs les dérivées du second ordre sont des grandeurs du même ordre que n et comme les dites dérivées se présentent linéairement dans les équations $R_{\mu\nu} = 0$ nous aurons pour une premiére approximation des termes du même ordre que n.

Cependant nous sommes plutôt intéressés au système (5.2) à (5.5) dans lequel les potentiels apparaissent avec leurs dérivées jusqu'au troisième ordre, ces dernières linéairement; ce qui veut dire que nous aurons des termes du même ordre de grandeur que n^2. Nous allons tout de suite nous assurer que ces termes sont les plus grands et que nous les avons tous en ne conservant que les dérivées du troisième ordre. Tout d'abord les coefficients a_{rs} et c^{-2} dans (5.2) sont au plus du même ordre de grandeur que n^0 ou "finis." D'un autre coté un simple examen de (5.6), (5.7) et des calculs intermédiaires nous montre qu'il nous reste à décider au sujet de termes tels que

$$\Gamma_{\alpha\alpha}{}^{\alpha}\bar{R}_{\alpha rs\sigma}, \qquad \Gamma_{0\tau}{}^{\alpha}\bar{R}_{0\alpha s\sigma}.$$

Or ces termes résultent du produit d'un facteur de grandeur "finie" comme $\Gamma_{\alpha\alpha}{}^{\alpha}$ avec un facteur d'ordre un comme

$$\bar{R}^{\alpha}{}_{rs\sigma} = \frac{\partial \Gamma_{r\sigma}{}^{\alpha}}{\partial x^s} - \frac{\partial \Gamma_{rs}{}^{\alpha}}{\partial x^{\sigma}} + \Gamma_{r\sigma}{}^{\alpha}\Gamma_{\alpha s}{}^{\alpha} - \Gamma_{rs}{}^{\alpha}\Gamma_{\alpha\sigma}{}^{\alpha}. \tag{6.1}$$

Nous en concluons que les dérivées covariantes par rapport aux deux métriques, celle de l'espace-temps d'une part et celle de $E(3)$ d'autre part ne diffèrent que par des termes négligeables en première approximation puis-qu'ils sont d'ordre un au plus. De même le second membre de (5.7) se réduit à son premier terme. Nous nous servirons de ces résultats pour transcrire le système (5.2) à (5.5) en fonction des tenseurs E_{rs} et H_{rs} et en cours de route, d'autres termes comme $c_0 R_{0rs\sigma}$ et $c_\sigma R_{0\alpha\beta\gamma}$ seront négligés toujours par suite des considérations précédentes. Nous emploierons le symbole $\backsim$ pour indiquer une égalité qui n'est vraie qu'en première approximation.

Partant d'abord de l'équation (5.2) on passe aux dérivées dans $E(3)$ et on multiplie ensuite les deux membres par $\epsilon^{\alpha\mu\nu}$; il en résulte (en tenant compte de $\bar{R}_{r0s\sigma} = -\bar{R}_{0rs\sigma}$

$$a^{\alpha\alpha}\bar{R}_{\alpha rs\sigma,\lambda}\,\epsilon^{\mu\nu\sigma} + c^{-2}\epsilon^{\mu\nu\sigma}\frac{\partial \bar{R}_{r0s\sigma}}{\partial t} \backsim 0.$$

Le premier terme peut être tranformé avec l'aide de (4.6); on obtient (puisque $\epsilon_{\alpha\mu\tau}$ joue le rôle d'une constante par rapport à la dérivation covariante)

$$\begin{aligned} a^{\alpha\alpha}\bar{R}_{\alpha rs\sigma,\lambda}\,\epsilon^{\mu\nu\sigma} &= a^{\alpha\alpha}E^{\alpha\beta}{}_{,\lambda}\,\epsilon_{\alpha\mu\tau}\epsilon_{\beta s\sigma}\epsilon^{\mu\nu\sigma} \\ &= 2\delta_{\beta}{}^{\mu}a^{\alpha\alpha}E^{\alpha\beta}{}_{,\lambda}\,\epsilon_{\alpha\mu\tau} \\ &= 2a^{\alpha\alpha}E^{\alpha\mu}{}_{,\lambda}\,\epsilon_{\alpha\mu\tau}. \end{aligned}$$

Par ailleurs, utilisant (4.11) et négligeant des termes tels que $c_0 H_r^{\ n}$ et

$$H_r^{\ n}\,\frac{\partial \epsilon_{nsv}}{\partial t}$$

on a

$$\epsilon^{msv}\,\frac{\partial \bar{R}_{r0sv}}{\partial t} \sim 2c\delta_n^{\ m}\,\frac{\partial H_r^{\ n}}{\partial t} \sim 2c\,\frac{\partial H_r^{\ m}}{\partial t}.$$

Alors notre équation peut s'écrire

$$a^{sn}E^{qm}{}_{,n}\,\epsilon_{qsr} + c^{-1}\,\frac{\partial H_r^{\ m}}{\partial t} \sim 0,$$

ou encore

$$E^{m}{}_{q,n}\,\epsilon^{qnr} + c^{-1}a^{ir}\,\frac{\partial H_i^{\ m}}{\partial t} \sim 0;$$

et finalement par suite de relations telles que

$$a^{ir}\,\frac{\partial H_i^{\ m}}{\partial t} \sim \frac{\partial H^{rm}}{\partial t},$$

on a

$$E_{mq,n}\,\epsilon^{qnr} + c^{-1}\,\frac{\partial H_m^{\ r}}{\partial t} \sim 0.$$

Transcrivons encore une fois ce résultat en y ajoutant les équations (5.3) à (5.5) transformées de la même façon, quoique beaucoup plus aisément avec l'aide de (4.8) et (4.11). On obtient pour le système au complet

$$\text{(6.2)}\qquad \begin{cases} E_{rq,s}\,\epsilon^{qsv} + \dfrac{1}{c}\,\dfrac{\partial H_r^{\ v}}{\partial t} \sim 0, \\ H_{rq,s}\,\epsilon^{qsv} - \dfrac{1}{c}\,\dfrac{\partial E_r^{\ v}}{\partial t} \sim 0, \\ E^{rs}{}_{,s} \sim 0, \\ H^{rs}{}_{,s} \sim 0. \end{cases}$$

Il est pour ainsi dire superflu de souligner ici qu'on retrouve les opérateurs vectoriels de la divergence et du rotationnel, dans les termes $E^{rs}{}_{,s}$ et $E_{rq,s}\epsilon^{qsv}$ de nos équations (6.2). En conséquence on reconnait aisément la structure des équations de Maxwell étendue aux tenseurs E_{rs} et H_{rs}.

Il importe maintenant de noter que dans les dérivées de second ordre $E_{rs,uv}$ les indices (uv) sont permutables en première approximation puisque, ainsi que nous allons le montrer à l'instant, la différence entre dérivées covariantes et dérivées ordinaires provient de termes qui tombent dans la partie négligeable. Effectivement nous avons vu que les éléments $E_{rs,u}$ sont du même ordre de grandeur que n^2 alors que les $\Gamma_{uv}{}^{r}$ sont finis comme n^0 de sorte que dans

$$E^{r}{}_{s,uv} = \frac{\partial E^{r}{}_{s,u}}{\partial x^{v}} + \Gamma_{uv}{}^{r}E^{n}{}_{s,u} - \Gamma_{sv}{}^{n}E^{r}{}_{n,u} - \Gamma_{uv}{}^{n}E^{r}{}_{s,n}$$

le premier terme au second membre prédomine avec une valeur de l'ordre de n^3. On a donc

$$E^r{}_{s,uv} \sim \frac{\partial E^r{}_{s,u}}{\partial x^v} \sim \frac{\partial^2 E^r{}_s}{\partial x^v \partial x^u},$$

d'où

$$E^r{}_{s,vu} \sim E^r{}_{s,uv}.$$

En conséquence on peut s'en rapporter aux calculs classiques pour résoudre le système (6.2) et caractériser les ondes au moyen de l'équation

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \sim 0. \tag{6.3}$$

Il est entendu ici que la fonction c intervenant dans les coefficients de nos équations (6.2) et (6.3) est considérée comme constante en première approximation et dans un voisinage qui peut contenir un certain nombre d'ondes (d'une part les potentiels oscillent avec des intensités qui sont de l'ordre de n^{-1} et d'autre part les variations de potentiel dues à la gravitation sont aussi très petites sauf au voisinage des singularités). Cette restriction a un voisinage est évidemment rendue nécessaire par la courbure du milieu de propagation mais elle n'en premet pas moins de reconnaître les caractéristiques essentielles des ondes électro-magnétiques dans ce nouvel aspect de la structure des équations de la gravitation.

7. Extension des résultats. Jusqu'à présent nos résultats sont limités au cas où les potentiels oscillent avec une amplitude de l'ordre de n^{-1}. Pour une amplitude plus grande, par exemple du même ordre que les $g_{\mu\nu}$, notre procédé d'approximation n'est plus valide car dans (6.1) les derniers termes sont du même ordre de grandeur que les dérivées secondes des premiers termes. En réalité, il n'y a pas lieu de considérer ce cas extrême pour lequel les potentiels changeraient périodiquement de signe.

Par contre nous allons montrer que l'onde reste maxwellienne pour des amplitudes aussi petites qu'on veut et indépendamment de la courbure du milieu de propagation. Considérons donc le cas où l'amplitude d'oscillation des potentiels est de l'ordre de n^{-r} avec $r > 1$ et posons $g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$ de telle façon que les $h_{\mu\nu}$ représentent en première approximation la partie oscillatoire de ces mêmes potentiels. Effectivement on peut toujours s'arranger pour que les $h_{\mu\nu}$ soient du même ordre de grandeur que l'amplitude d'oscillation, à savoir de l'ordre de n^{-r}, et que par ailleurs les $\gamma_{\mu\nu}$ oscillent avec des amplitudes $< n^{-r-1}$.

Notons alors les résultats suivants:

7.1. Les dérivées d'ordre k des $h_{\mu\nu}$ oscillent avec des amplitudes de l'ordre de n^{-r+k} et l'ordre de grandeur de ces quantités ne dépasse pas celui de leur amplitude d'oscillation.

7.2. Les $\gamma_{\mu\nu}$ et leurs dérivées jusqu'au deuxième ordre sont des grandeurs "finies" au plus.

Disons maintenant que sans nous inquiéter pour le moment de savoir à quoi cela conduit nous allons séparer les équations (5.1) de façon à ne garder que les termes oscillant avec la plus grande intensité. Nous allons voir tout de suite que nous avons tous ces termes en ne gardant que ceux qui contiennent les dérivées du troisième ordre des $h_{\mu\nu}$ et qu'ainsi l'intensité d'oscillation sera de l'ordre de n^{-r+3}. Effectivement un simple examen de (5.6), (5.7) et (6.1) nous montre que dans les termes ne contenant pas de dérivées du troisième ordre seulement *un* des facteurs peut être une dérivée seconde des $h_{\mu\nu}$ et que tous les autres facteurs sont des grandeurs *finies* au plus de sorte que les termes en question oscillent à l'ordre $-r+2$ au plus.

Par ailleurs on peut effectuer le développement de (5.1) en introduisant d'abord les $E_{\tau\alpha,\beta}$ et $H_{\tau\alpha,\beta}$ comme dans la section précédente et, puisque les dérivées du troisième ordre interviennent linéairement, on peut dédoubler ces tenseurs d'une façon qu'on peut représenter symboliquement par

$$E_{\tau\alpha,\beta} = E_{\tau\alpha,\beta}(\gamma) + E_{\tau\alpha,\beta}(h)$$

pour ne garder que la partie oscillant avec la plus grande intensité, soit à l'ordre $-r+3$. Les $E_{\tau\alpha,\beta}(h)$ et $H_{\tau\alpha,\beta}(h)$ sont alors des tenseurs pour lesquels la métrique est déterminée en première approximation par les valeurs moyennes $\gamma_{\mu\nu}$.

Bien entendu les termes que nous voulons maintenant laisser de coté ne sont pas nécessairement plus petits que les autres (c'est plutôt le contraire pour $r > 2$), cependant si les équations (5.2) à (5.5) doivent être vérifiées il faut que les termes de la partie oscillatoire se compensent entre eux en première approximation et que leur somme (dans chaque équation) se réduise à une quantité n'oscillant plus qu'à l'ordre $-r+2$. Enfin l'ordre de grandeur de ces dernières quantités (ou sommes) ne dépassera pas $-r+2$ si les $h_{\mu\nu}$ sont convenablement choisis. En effet, il n'y a là qu'une question de partie constante en première approximation qui peut être renvoyée dans les $\gamma_{\mu\nu}$. Bref on est de nouveau conduit aux équations (6.2) pour la partie oscillatoire et les ondes de petite amplitude possèdent toujours les caractéristiques que comporte la structure maxwellienne.

Il est encore possible d'étendre nos résultats au cas où la fréquence n'est pas nécessairement très élevée mais cette fois nous serons limités à un milieu de propagation presqu'euclidien. Posons donc

$$c^2 = 1 + h_{00}, \qquad g_{\tau\alpha} = -\delta_\tau^\alpha - h_{\tau\alpha},$$

où les $h_{\tau\alpha}$ et h_{00} sont des quantités élémentaires du premier ordre. Nous avons donc recours ici à un procédé classique d'approximation qui a permis, en particulier, de caractériser les ondes dites gravitationnelles au moyen de l'équation (6.3). Cependant l'introduction de nos tenseurs $E_{\tau\alpha}$ et $H_{\tau\alpha}$ va nous permettre d'apporter un élément nouveau à ce résultat.

Dans une première approximation suivant cette méthode on ne conserve que les termes linéaires. Alors le lecteur n'aura aucune peine à s'assurer que les équations (5.1) se réduisent aux dérivées de troisème ordre des $h_{\mu\nu}$ et qu'ainsi

on retrouve encore le système (6.2) avec la différence que les dérivées covariantes sont remplacées par les dérivées ordinaires. Nous en déduisons que dans le domaine des fréquences de la radio nos ondes sont encore maxwelliennes pourvu que la courbure du milieu de propagation soit assez petite.

Ce n'est pas sans raison que nous faisons ici la distinction pour exclure de nos déductions les ondes gravitationnelles proprement dites. En effet dans ce cas la fréquence prend des valeurs extrêmement petites (de l'ordre de 10^{-18} dans le cas du mouvement de la terre autour du soleil et avec l'unité de temps comme plus haut) ce qui introduit dans le problème un aspect critique que nous ne voulons pas discuter ici.

8. Conclusion. Ainsi donc la structure des équations de Maxwell se retrouve en première approximation et comme conséquence directe des équations $\bar{R}_{\mu\nu} = 0$ pour caractériser dans presque tous les cas les ondes qui pourraient exister dans le continuum espace-temps de la relativité. En résumé si la fréquence prend des valeurs correspondant à l'infra-rouge ou plus grandes nos ondes sont caractérisées par les équations (6.2) et (6.3) indépendamment de la courbure du milieu; par contre si la fréquence est moins élevée nos équations restent valides seulement si la courbure est petite. Notons encore une fois qu'en dernière analyse ce sont les éléments de courbure $\bar{R}_{\alpha\beta\mu\nu}$ qui vérifient de telles équations et que cela est rendu possible simplement par la façon, à tout le moins remarquable, suivant laquelle les dits éléments se laissent regrouper en deux tenseurs E_{rs} at H_{rs} jouant le rôle des vecteurs **E** et **H** des équations de Maxwell.

En conséquence l'invariance du phénomène ne fait pas de doute car les oscillations qu'on imposerait arbitrairement au réseau de coordonneés ne pourraient être caractérisées par des équations invariantes. Nous avons donc bien une onde et nous faisons tout-de-suite le rapprochement avec les ondes électromagnétiques. Or ces dernières se laissent représenter au moyen de vecteurs, du moins dans les aspects que nous leur connaissons, alors que pour les nouvelles ondes les tenseurs E_{rs} et H_{rs} présentent tout de même une structure plus complexe et laissent entrevoir de grandes possibilités.

Pouvons nous dire maintenant que, dans le continuum $\bar{R}_{\mu\nu} = 0$, nous avons identifié les ondes électro-magnétiques sous une forme nouvelle dont nous ne connaissions que quelques aspects? Il faut avouer que pour le moment tout essai d'interprétation dans ce sens pose des problèmes dont la solution ne semble pas encore à notre portée. Tout d'abord les ondes que nous venons de caractériser existent-elles? Ou, plus précisément, existe-t-il des solutions de ce genre qui soient régulières à l'infini? Voilà une question qui nous laisse perplexes devant la complexité des équations $\bar{R}_{\mu\nu} = 0$. Et pourtant il serait du plus haut intérêt de savoir déterminer les possibilités de ces équations en ce qui concerne des singularités accompagnées d'un champ oscillatoire stationnaire ou irradiant de l'énergie, et qui entrainerait sans doute un phénomène de quantification. Quoiqu'il en soit nous terminons ce travail en exposant deux autres résultats qui prolongent encore l'analogie avec les équations de l'électro-magnétisme.

Note A. Dans cette section nous nous occuperons de deux invariants de l'espace-temps et, bien entendu toujours dans le cas de $g_{0\gamma} = 0$, nous trouverons moyen de les exprimer en fonction des tenseurs $E_{\gamma\delta}$ et $H_{\gamma\delta}$.

Tout d'abord on vérifie sans peine:

$$\begin{aligned}\bar{R}_{\alpha\beta\mu\nu}\bar{R}^{\alpha\beta\mu\nu} &= \bar{R}_{\gamma\sigma\mu\nu}\bar{R}^{\gamma\sigma\mu\nu} + 2\bar{R}_{\gamma 0\mu\nu}\bar{R}^{\gamma 0\mu\nu} \\ &= \bar{R}_{\gamma\sigma\mu\nu}\bar{R}^{\gamma\sigma\mu\nu} + 2\bar{R}_{\gamma\sigma 0\nu}\bar{R}^{\gamma\sigma 0\nu} + 2\bar{R}_{\gamma 0\mu\nu}\bar{R}^{\gamma 0\mu\nu} + 4\bar{R}_{\gamma 00\nu}\bar{R}^{\gamma 00\nu} \\ &= \bar{R}_{\gamma\sigma\mu\nu}\bar{R}^{\gamma\sigma\mu\nu} + 4\bar{R}_{\gamma\sigma 0\nu}\bar{R}^{\gamma\sigma 0\nu} + 4\bar{R}_{\gamma 00\nu}\bar{R}^{\gamma 00\nu}.\end{aligned} \tag{A.1}$$

D'un autre coté, avec l'aide de (2.1a) et (4.8) on a

$$\bar{R}^{\gamma 00\nu} = g^{\gamma\alpha}g^{00}g^{0\mu}g^{\sigma\nu}\bar{R}_{\alpha\beta\mu\sigma} = c^{-4}a^{\gamma\beta}a^{\mu\nu}\bar{R}_{\beta 00\mu} = c^{-2}E^{\gamma\nu},$$

d'où

$$\bar{R}_{\gamma 00\nu}\bar{R}^{\gamma 00\nu} = E_{\gamma\nu}E^{\gamma\nu}. \tag{A.2}$$

Puis étant donné que $\bar{R}_{\mu\nu\sigma\tau}$ est un tenseur dans $E(3)$ la relation (4.3):

$$4E^{\gamma\nu} = \bar{R}_{\mu\nu\sigma\tau}\epsilon^{\mu\sigma\gamma}\epsilon^{\nu\sigma\tau}$$

entraine (compte tenu de (2.1a) comme précédemment

$$4E_{\gamma\nu} = \bar{R}^{\mu\nu\sigma\tau}\epsilon_{\mu\sigma\gamma}\epsilon_{\nu\sigma\tau},$$

d'où

$$16E_{\gamma\nu}E^{\gamma\nu} = \bar{R}^{\mu\nu\sigma\tau}\bar{R}_{\alpha\beta\gamma\delta}\delta_{\mu\nu}^{\alpha\beta}\delta_{\sigma\tau}^{\gamma\delta} = 4\bar{R}_{\mu\nu\sigma\tau}\bar{R}^{\mu\nu\sigma\tau}. \tag{A.3}$$

Rappelons maintenant (4.4):

$$2cH_{\gamma}^{\ \sigma} = \bar{R}_{\gamma 0\nu\sigma}\epsilon^{\mu\nu\sigma}.$$

Nous avons aussi comme pour les relations précédentes et tenant compte de (2.1a)

$$2cH^{\gamma}_{\ \sigma} = -c^2\bar{R}^{\gamma 0\mu\nu}\epsilon_{\mu\nu\sigma}.$$

Enfin par contraction il vient

$$4c^2H_{\gamma}^{\ \sigma}H^{\gamma}_{\ \sigma} = -c^2\bar{R}_{\gamma 0\mu\nu}\bar{R}^{\gamma 0\mu\nu}\delta_{\mu\nu}^{\ \ \mu\nu}$$

ou

$$4H_{\gamma\sigma}H^{\gamma\sigma} = -2\bar{R}_{\gamma 0\mu\nu}\bar{R}^{\gamma 0\mu\nu}. \tag{A.4}$$

Alors les relations (A.2) à (A.4) permettent d'écrire (A.1) sous la forme

$$\bar{R}_{\alpha\beta\mu\nu}\bar{R}^{\alpha\beta\mu\nu} = 8(E_{\gamma\nu}E^{\gamma\nu} - H_{\gamma\nu}H^{\gamma\nu}). \tag{A.5}$$

Nous ferons de même avec un autre invariant dont l'expression fait intervenir le tenseur universel suivant

$$\epsilon_{\alpha\beta\gamma\delta} = g^{\frac{1}{2}}e_{\alpha\beta\gamma\delta},$$

avec $\epsilon_{\alpha\beta\gamma\delta} = 0$ ou ± 1 d'après la règle:

0 si deux quelconques des indices $\alpha\beta\gamma\delta$ ont la même valeur
1 si $\alpha\beta\gamma\delta$ est une permutation paire nes nombres 0, 1, 2, 3.
-1 si $\alpha\beta\gamma\delta$ est une permutation impaire des mêmes nombres.

L'ensemble des composantes covariantes non nulles de ce tenseur peuvent aisément être représentées au moyen d'un tenseur de E(3) comme il a été expliqué dans §3. Le nouveau tenseur sera de troisième ordre puisqu'un des quatre indices $\alpha\beta\gamma\delta$ doit prendre la valeur zéro. On notera d'abord que les permutations $0rst$ et rst ont la même parité (c'est-à-dire le même signe) par rapport aux nombres 0, 1, 2, 3, et 1, 2, 3 respectivement. On a d'ailleurs $g^{\frac{1}{2}} = ica^{\frac{1}{2}}$ avec $i = (-1)^{\frac{1}{2}}$ de sorte que

$$\epsilon_{0rst} = ic\epsilon_{rst}.$$

Alors on a successivement

$$\begin{aligned}
\bar{R}^{\alpha\beta\mu\nu}\bar{R}_{\alpha\beta}{}^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} &= \bar{R}^{mn\mu\nu}\bar{R}_{mn}{}^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} + 2\bar{R}^{n0\mu\nu}\bar{R}_{n0}{}^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} \\
&= 2\bar{R}^{mn\mu\nu}\bar{R}_{mn}{}^{0s}\epsilon_{\mu\nu0s} + 2\bar{R}^{mn0\nu}\bar{R}_{mn}{}^{rs}\epsilon_{0\nu rs} \\
&\qquad + 4\bar{R}^{n0\mu\nu}\bar{R}_{n0}{}^{0s}\epsilon_{\mu\nu0s} + 4\bar{R}^{n00\nu}\bar{R}_{n0}{}^{rs}\epsilon_{0\nu rs} \\
&= 8\bar{R}^{n0\mu\nu}\bar{R}_{n0}{}^{0s}\epsilon_{0\mu\nu s} + 4\bar{R}^{mn\mu\nu}\bar{R}_{mn}{}^{0s}\epsilon_{0\mu\nu s} \\
&= 4ic(2\bar{R}^{n0\mu\nu}\bar{R}_{n0}{}^{0s}\epsilon_{\mu\nu s} + \bar{R}^{mn\mu\nu}\bar{R}_{mn}{}^{0s}\epsilon_{\mu\nu s}).
\end{aligned}$$

Les deux termes de la parenthèse se retrouvent à un facteur près par la contraction de E_{rs} avec H^{rs} de deux façons différentes à partir de (4.3), (4.4) et (4.8). Les calculs sont analogues à ceux qui ont donné (A.2) à (A.4) et entrainent finalement

$$\bar{R}^{\alpha\beta\mu\nu}\bar{R}_{\alpha\beta}{}^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = 32iE_{rs}H^{rs}. \tag{A.6}$$

Il faut remarquer que les relations (A.5) et (A.6) sont possibles en autant que la forme (1.1) de notre élément d'arc est conservée. Cependant les invariants au premier membre nous permettent de conclure que les fonctions

$$E_{rs}E^{rs} - H_{rs}H^{rs}, \quad E_{rs}H^{rs}$$

sont elles-mêmes invariantes par rapport à la transformation de Lorentz. On retrouve ainsi une propriété que la forme relativiste des équations électromagnétiques confère aux invariants

$$\mathbf{E}^2 - \mathbf{H}^2, \quad \mathbf{E}\cdot\mathbf{H}.$$

Note B. Partant de (6.1) nous ferons $r = 0$ et nous séparerons les sommes relatives à l'indice a.
Alors

$$\bar{R}^{a}{}_{0s0} = \frac{\partial\Gamma_{0s}{}^{a}}{\partial x^{0}} - \frac{\partial\Gamma_{00}{}^{a}}{\partial x^{s}} + \Gamma_{0s}{}^{t}\Gamma_{t0}{}^{a} - \Gamma_{00}{}^{t}\Gamma_{ts}{}^{a} + \Gamma_{0s}{}^{0}\Gamma_{00}{}^{a} - \Gamma_{00}{}^{0}\Gamma_{0s}{}^{a}.$$

Puis avec l'aide de (2.3),

$$\bar{R}^{n}{}_{0sv} = \frac{\partial(cA^{n}{}_{v})}{\partial x^{s}} - \frac{\partial(cA^{n}{}_{s})}{\partial x^{v}} + c(\Gamma_{is}{}^{n}A^{i}{}_{v} - \Gamma_{iv}{}^{n}A^{i}{}_{s}) + c_{s}A^{n}{}_{v} - c_{v}A^{n}{}_{s}$$

$$= c\left(\frac{\partial A^{n}{}_{v}}{\partial x^{s}} - \frac{\partial A^{n}{}_{s}}{\partial x^{v}}\right) + c(\Gamma_{is}{}^{n}A^{i}{}_{v} - \Gamma_{iv}{}^{n}A^{i}{}_{s}).$$

Ajoutons à ce dernier résultat le terme identiquement nul

$$c(-\Gamma_{sv}{}^{i}A^{n}{}_{i} + \Gamma_{sv}{}^{i}A^{n}{}_{i}),$$

et nous reconnaitrons immédiatement le développement de deux dérivées covariantes de sorte que

(B.1) $$\bar{R}^{n}{}_{0sv} = c(A^{n}{}_{v,s} - A^{n}{}_{s,v}).$$

Pour $v = n$ on retrouve les trois équations $\bar{R}_{0s} = 0$ sous la forme

(B.2) $$A^{n}{}_{n,s} - A^{n}{}_{s,n} = 0.$$

D'ailleurs en tenant compte de (2.1) l'équation (B.1) peut s'écrire

$$\bar{R}_{n0sv} = c(A_{ns,v} - A_{nv,s}),$$

d'où, après multiplication des deux membres par ϵ^{svr} et substitution au moyen de (4.4)

$$2cH_{n}{}^{r} = c(A_{ns,v} - A_{nv,s})\epsilon^{svr} = 2cA_{ns,v}\,\epsilon^{svr}.$$

On a donc finalement

(B.3) $$H_{n}{}^{r} = A_{ns,v}\,\epsilon^{svr}$$

et on retrouve la forme de la relation classique $\mathbf{H} = \mathbf{rot}\ \mathbf{A}$ étendue au tenseur A_{rs}.

Université Laval

THE COORDINATE CONDITIONS AND THE EQUATIONS OF MOTION

L. INFELD

1. Introduction. The problem of the field equations and the equations of motion in general relativity theory is now sufficiently clarified. The equations of motion can be deduced from pure field equations by treating matter as singularities, [2; 3], or from field equations with the energy momentum tensor [4]. Recently two papers appeared in which the problem of the coordinate system was considered [5; 8]. The two papers are in general agreement as far as the role of the coordinate system is concerned. Yet there are some differences which require clarification.

For this purpose it will be useful to analyse a very simple and well-known example that will emphasize some of the essential ideas by which field and motion are connected. The problem of motion is complicated in the case of two bodies but much simpler in the case of one body, when we may use the geodetic principle, which, by the way, can also be deduced from the field equations [1; 6]. Therefore, we shall begin by returning to this rather elementary problem and analyse some of its features, which appear in a changed form in the two-body problem. An interesting by-product of this work is a method of deducing perihelion motion, which seems to me simpler and more instructive than any others I know.

2. The Schwarzschild solution and the change of the coordinate system. It is usual to write the quadratic form of the Schwarzschild solution in the following way:

$$(2.1) \qquad ds = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

What are the characteristic features of this coordinate system? For large r the only difference between the Galilean and the Schwarzschild field consists of additional terms that are essentially the Newtonian potentials. For $r \to \infty$ the field becomes Galilean. As a rule it is in such a coordinate system that we calculate the perihelian motion. We ask: Is it possible to find a coordinate system in which the equations of the path are the ordinary equations of a conic without any perihelion motion? This does not mean that the perihelion motion can be wiped out. But it does mean that we can introduce an artificial observer moving with the angular velocity of the perihelion motion.

To show this, at least for the *mean* path, it is sufficient to introduce a change in one variable, that is ϕ. We write:

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$$\phi = \psi + \omega(\psi). \tag{2.2}$$

We shall assume not that $\omega(\psi)$ is small, but that its derivatives are small, that is we shall consistently omit all expressions of higher order than $d\omega/d\psi$. This is our only assumption.

Thus from (2.2) it follows that:

$$d\phi^2 = d\psi^2\left(1 + 2\frac{d\omega}{d\psi}\right).$$

Therefore, in our new coordinate system, the metrical form (2.1) takes the following form:

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\psi^2\left(1 + 2\frac{d\omega}{d\psi}\right). \tag{2.3}$$

The next step is to calculate the differential equations of the geodetic line. This calculation is absolutely straightforward and little difficulty is added by taking (2.3) instead of (2.1) We begin with the procedure described extensively in any book on general relativity theory and take $\theta = \frac{1}{2}\pi$. We then obtain three equations. The first is simply (2.1) or, in our coordinate system, (2.3) for $\theta = \frac{1}{2}\pi$ and $d\theta = 0$. Then we have two integrals. One of them which represents the conservation of momenta is, because of (2.2),

$$r^2\frac{d\psi}{ds} + r^2\frac{d\omega}{ds} = h. \tag{2.4}$$

The third equation, also integrated, is

$$\frac{dt}{ds} = C\left(1 - \frac{2m}{r}\right)^{-1} \tag{2.5}$$

or precisely the same as in the original coordinate system represented by (2.1). Thus (2.3), (2.4), (2.5) represent our equations of motion.

We can write (2.4):

$$r^2\frac{d\psi}{ds} + r^2\frac{d\omega}{d\psi}\frac{d\psi}{ds} = h, \tag{2.4a}$$

therefore:

$$\frac{d\psi}{ds} = \frac{h}{r^2} - \frac{d\omega}{d\psi}\frac{d\psi}{ds}.$$

Thus, neglecting higher powers of $d\omega/d\psi$, we can write (2.4):

$$\frac{d\psi}{ds} = \frac{h}{r^2}\left(1 - \frac{d\omega}{d\psi}\right). \tag{2.6}$$

Besides this, we have equations (2.5) and (2.3) for $d\theta = 0$, $\theta = \frac{1}{2}\pi$, which we shall rewrite now:

$$\text{(2.3a)}\quad \left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2-\left(\frac{dr}{ds}\right)^2\left(1-\frac{2m}{r}\right)^{-1}-r^2\left(\frac{d\psi}{ds}\right)^2\left(1+2\frac{d\omega}{d\psi}\right)=1.$$

Here we replace:

$$\frac{dt}{ds}\quad\text{by}\quad C\left(1-\frac{2m}{r}\right)^{-1},\qquad\text{according to (2.5),}$$

$$\frac{dr}{ds}\quad\text{by}\quad\frac{dr}{d\psi}\frac{d\psi}{ds},$$

$$\frac{d\psi}{ds}\quad\text{by}\quad\frac{h}{r^2}\left(1-\frac{d\omega}{d\psi}\right),\qquad\text{according to (2.6).}$$

We obtain

$$\text{(2.7)}\qquad C^2-\left(\frac{dr}{d\psi}\right)^2\left(1-2\frac{d\omega}{d\psi}\right)\frac{h^2}{r^4}-\frac{h^2}{r^2}\left(1-\frac{2m}{r}\right)=1-\frac{2m}{r}.$$

We could have done this still more simply by taking the usual equation in the coordinate system (2.1), that is the equation:

$$C^2-\left(\frac{dr}{d\phi}\right)^2\frac{h^2}{r^4}-\frac{h^2}{r^2}\left(1-\frac{2m}{r}\right)=1-\frac{2m}{r}$$

and introducing into this equation the transformation (2.2). Equation (2.7) follows immediately.

Now we write (2.7) in the following form, introducing as usual $u=r^{-1}$:

$$\text{(2.8)}\quad C^2-h^2\left(\frac{du}{d\psi}\right)^2-h^2u^2-1+2mu+2h^2\left(\frac{du}{d\psi}\right)^2\left(\frac{d\omega}{d\psi}\right)+2mh^2u^3=0.$$

Our question is whether by a proper choice of the function ω we can rigorously solve the equation (2.8) by

$$\text{(2.9)}\qquad u=\frac{1+e\cos\psi}{p}.$$

This is indeed possible and to show it we split the equation (2.8) into two parts which we shall satisfy separately:

$$\text{(2.10)}\qquad C'^2-h^2\left(\frac{du}{d\psi}\right)^2-h^2u^2-1+2mu=0,$$

$$\text{(2.11)}\qquad C^2-C'^2+2h^2\left(\frac{du}{d\psi}\right)^2\left(\frac{d\omega}{d\psi}\right)+2mh^2u^3=0.$$

Here (2.10) is the Newtonian equation, in which, provisionally, the constant C' may be regarded as unknown. Its solution therefore is (2.9), where e and p are functions of C' and h. Now as to the equation (2.11), we introduce into it (2.9), regarding it as an equation for ω and C'. We obtain from (2.11):

$$\text{(2.11a)}\quad e^2\left(\frac{d\omega}{d\psi}\right)\sin^2\psi+\frac{m}{p}(1+3e\cos\psi+3e^2\cos^2\psi+e^3\cos^3\psi)+\frac{(C^2-C'^2)p^2}{2h^2}=0$$

or, assuming $\sin^2\psi \neq 0$:

$$\text{(2.11b)}\qquad \frac{d\omega}{d\psi}+\frac{3m}{pe}\frac{\cos\psi}{\sin^2\psi}-\frac{3m}{p}+\frac{me}{p}\frac{\cos^3\psi}{\sin^2\psi}+\frac{1}{\sin^2\psi}\left(\frac{m}{pe^2}+\frac{3m}{p}+\frac{C^2-C'^2}{2h^2}p^2\right)=0.$$

Thus we can solve this equation, by putting:

$$\text{(2.12)}\qquad C'^2=C^2+\frac{2mh^2}{p^3e^2}+\frac{6mh^2}{p^3}$$

and

$$\text{(2.13)}\qquad \frac{d\omega}{d\psi}+\frac{3m\cos\psi}{pe\sin^2\psi}-\frac{3m}{p}+\frac{me\cos^3\psi}{p\sin^2\psi}=0.$$

Equation (2.12) says that the ellipse is slightly changed from the one which corresponds to the constant C. Remembering that

$$h^2 \text{ is of the order } r^4\omega^2$$

(r being the "distance" of the planet from the sun) and that

$$p \text{ is of the order of } r,$$

we see that

$$C^2-C'^2 \text{ is of the order } mv^2/r$$

(v being the "velocity"). Therefore (2.10) in which C^2 was replaced by C'^2 gives the "relativistic" correction to the "energy."

Equation (2.13) is not sufficient to determine ω in the neighbourhood of $2\pi n$. But it is sufficient to determine the perihelion motion. We have:

$$\phi=\psi+\omega(\psi)=f(\psi).$$

Let us write

$$\phi_n = f(2\pi n)=2\pi n+\omega_n$$

$$\phi_{n+1}=f(2\pi n+2\pi)=2\pi(n+1)+\omega_{n+1},$$

then the definition of the perihelion motion is:

$$\phi_{n+1}-\phi_n-2\pi=\omega_{n+1}-\omega_n.$$

Now integrating (2.13) from $2\pi n$ to $2\pi(n+1)$ we have

$$\omega_{n+1}-\omega_n=6m\pi/p,$$

which formula expresses the famous perihelion motion.

In order that the perihelion motion describe reality, it must be, in a certain sense at least, independent of the coordinate system. Indeed, the perihelion motion *is* independent of the coordinate system, as long as the coordinate system is Galilean at infinity.

The description of the perihelion motion of, say, Mercury is based on the assumption that the observer is so far away and in such a coordinate system

that *his* field is Galilean. Otherwise, nothing else is required from the coordinate system. The one represented by (2.1) would do just as well as any other as long as it is Galilean at infinity.

Imagine an observer (in a Galilean local coordinate system) very far from the "sun," pointing with one rigid rod towards the sun and with another towards Mercury, at a moment when the angle between the two rods is smallest. In principle, this can be done, since, in a Galilean coordinate system, all the concepts—rigid rods, angles, etc., are the familiar Euclidean concepts. Then the rigid rod pointing toward Mercury will describe an ellipse. But after one rotation, the perihelion position (that is, the minimum angle between the two rods) will be slightly shifted. After very many rotations, the original perihelion position from which the idealized observer started will be reached again. We can ask: After how many rotations of Mercury will the perihelion point complete *one* rotation? Such a formulation of the question has nothing to do with any special coordinate system but it does assume that the chosen coordinate system is Galilean at infinity, because only in such a coordinate system can we use freely the concepts of rigid body, angles, etc. Thus we can have either Newtonian motion and a non-Galilean field at infinity, or perihelion motion and a Galilean field at infinity.

3. The two-body problem. The above discussion should help to clarify the problem of the coordinate system for a two-body problem. It was solved in a series of papers, not by assuming the principle of a geodetic line but by deducing the equations of motion from the field equations. One of the essential ideas used then is the "new approximation method." By its use we find the field in the second and third approximations. Then from this knowledge of the field we deduce the equations of motion in the fourth approximation. These are the Newtonian equations of motion. The next step is to find the field in the fourth and fifth approximations. Finally, we deduce the equations of motion up to the sixth approximation. These equations give us the perihelion motion of the two (or many) body problem. It would be extremely difficult, technically, to go beyond the sixth approximation. But it is not necessary to do so. All contributions to the motion in higher than the sixth approximation can be wiped out by a proper choice of a coordinate system [5].

Thus having described the procedure in general terms we shall write down the field equations in the second and third approximations [3, p. 219]. These are:

$$\underset{2}{\gamma}_{00,ss} = 0 \tag{3.1}$$

$$-\underset{3}{\gamma}_{0m,ss} + \underset{3}{\gamma}_{0s,ms} = \underset{2}{\gamma}_{00,0m}; \qquad m, s = 1, 2, 3,$$

where

$$\gamma_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}h_{\sigma\rho} \qquad \mu, \nu, \sigma, \rho = 0, 1, 2, 3$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\eta_{ms} = -\delta_{ms}; \quad \eta_{00} = 1; \quad \eta_{0m} = 0.$$

Here, and later, Greek indices take the values 0, 1, 2, 3 and Latin indices the values 1, 2, 3. One more remark: the stroke means differentiation:

$$\gamma_{\mu\nu/s} = \frac{\partial g_{\mu\nu}}{\partial x^s} = g_{\mu\nu,s}$$

$$\gamma_{\mu\nu/0} = \frac{\partial g_{\mu\nu}}{\partial x^0} = \frac{\partial g_{\mu\nu}}{\partial \tau}\lambda = \lambda g_{\mu\nu,0} \qquad (\lambda x^0 = \tau).$$

Thus the differentiation with respect to τ raises the order, since λ is the parameter of the development and its order is indicated by a number below the γ's.

The essential point in the use of the "new approximation method" consisted in the fact that we began the development of γ_{00}, γ_{0m}, and γ_{mn} with $\underset{2}{\gamma}_{00}$, $\underset{3}{\gamma}_{0m}$, and $\underset{4}{\gamma}_{mn}$ respectively:

$$\begin{aligned} \gamma_{00} &= \lambda^2\underset{2}{\gamma}_{00} + \lambda^4\underset{4}{\gamma}_{00} + \dots \\ \gamma_{0m} &= \lambda^3\underset{3}{\gamma}_{0m} + \lambda^5\underset{5}{\gamma}_{0m} + \dots \\ \gamma_{mn} &= \lambda^4\underset{4}{\gamma}_{mn} + \lambda^6\underset{6}{\gamma}_{mn} + \dots . \end{aligned} \tag{3.2}$$

If we look at (3.1) we see that γ_{00} must be a harmonic function. Let us choose an harmonic function that represents two moving singularities. But then γ_{0m} is not uniquely determined. Indeed let us replace

$$\underset{3}{\gamma}_{0m} \quad \text{by} \quad \underset{3}{\gamma}_{0m} + a_{0,m}$$

a_0 being an arbitrary function. Then $\underset{3}{\gamma}_{0m} + a_{0,m}$ *is* a solution. Thus we could ask whether the arbitrary function $\underset{3}{a}_{0,m}$ can have any influence upon the equations of motion. It certainly does not have any influence on the equations of motion of the fourth order (which are the Newtonian equations of motion). This follows from a theorem proved before [3, p. 233]. But a straightforward, though troublesome, calculation [9] shows that it has no influence, either, upon the equations of motion up to the sixth order. Thus it is not true [8] as Papapetrou claims that we could have changed the equations of motion by changing the coordinate condition involving $\underset{2}{\gamma}_{00}$ and $\underset{3}{\gamma}_{0m}$. Thus nothing can change the equations of motion up to the sixth order, as long as we stick to the following procedures:

1. To introduce solutions of Laplace's equations for $\underset{2}{\gamma}_{00}$ that represent the Newtonian fields.

2. To use the "new approximation method", starting with $\underset{2}{\gamma}_{00}$, $\underset{3}{\gamma}_{0m}$, $\underset{4}{\gamma}_{mn}$ respectively.

Once these procedures are assumed and no singularities later arbitrarily introduced, the equations of motion up to the sixth order are uniquely determined, giving us, in consequence, the perihelion motion of a double star.

But we have seen in the first part, that we can wipe out the perihelion motion by a proper choice of the coordinate system. There seems to be a contradiction between our statement here and the result of the first part of this paper. The solution is simple enough: to introduce a coordinate system that would wipe out the perihelion motion we would have to violate the second of our assumptions just quoted.

Although the corresponding calculation is elementary, we shall show it in detail in order to clarify the problem.

Let us rewrite (2.1) in the form:

$$(3.3) \qquad ds^2 = \left(1 - \frac{2m}{\rho}\right)dt^2 - \left(1 - \frac{2m}{\rho}\right)^{-1} d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

We introduce the transformation

$$(3.4) \qquad \rho = r + m = r + \underset{2}{m},$$

since m is of the second order. Again we rewrite (3.3) keeping only expressions up to the second order:

$$(3.5) \quad ds^2 = (1 - \underset{2}{2mr^{-1}})dt^2 - dr^2(1 + \underset{2}{2mr^{-1}})((dx^1)^2 + (dx^2)^2 + (dx^3)^2).$$

(Here the connection between r, θ, ϕ and x^1, x^2, x^3 is obvious.)

Thus we have in (3.5)

$$\underset{2}{h_{00}} = -\underset{2}{2mr^{-1}}; \quad \underset{2}{h_{mn}} = -\delta_{mn}\underset{2}{2mr^{-1}}.$$

Therefore:

$$\underset{2}{\gamma_{00}} = \tfrac{1}{2}h_{00} + \tfrac{1}{2}h_{ss} = -\underset{2}{4mr^{-1}},$$

$$\underset{2}{\gamma_{0m}} = 0,$$

$$\underset{2}{\gamma_{mn}} = h_{mn} - \tfrac{1}{2}\delta_{mn}h_{ss} + \tfrac{1}{2}\delta_{mn}h_{00} = 0.$$

Thus this solution satisfies our conditions, because $\underset{2}{\gamma_{mn}} = 0$ and because $\underset{2}{\gamma_{00}}$ represents a Newtonian field belonging to a particle.

To wipe out the perihelion motion we would have to introduce a rotation in the "plane" of the motion with an angular velocity ω chosen so as to wipe out such a motion. Thus we introduce the transformation:

$$x^1 = \xi^1\cos\omega - \xi^2\sin\omega$$

$$x^2 = \xi^1\sin\omega + \xi^2\cos\omega$$

$$x^3 = \xi^3.$$

In this coordinate system we have the following quadratic form:

$$(3.6)\qquad ds^2 = [1 - 2mr^{-1} + \dot{\omega}^2((\xi^1)^2 + (\xi^2)^2)]dt^2 + 2\xi^1\dot{\omega}dt\,d\xi^2 - 2\xi^2\dot{\omega}dt\,d\xi^1 - dr^2(1 + 2mr^{-1})((d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2).$$

If $\dot{\omega} = \underset{1}{\dot{\omega}}$ is of order one, we see that our quadratic form violates both the conditions on which our approximation procedure is built. First, $\underset{2}{\gamma_{00}}$ does not vanish for $r \to \infty$, and does not represent a Newtonian field of a particle. Secondly, $\underset{3}{\gamma_{0m}}$ does not start with $\underset{3}{\gamma_{0m}}$ but with $\underset{1}{\gamma_{0m}}$. Thirdly,

$$\underset{2}{\gamma_{mn}} = \tfrac{1}{2}\dot{\omega}^2((\xi^1)^2 + (\xi^2)^2)\,\delta_{mn},$$

that is, $\underset{2}{\gamma_{mn}}$ does not start with $\underset{4}{\gamma_{mn}}$.

Thus, returning to the two-body problem, we see that the conditions of our method are sufficiently stringent to insure uniquely the equations of motion up to the sixth order without any additional conditions concerning the coordinate system.

Therefore the choice of a coordinate system has no influence upon the equations of motion up to the sixth order. A straightforward calculation shows that this is true for $\underset{3}{\gamma_{0m}}$; that the addition of an arbitrary function $a_{0,m}$ does not change the surface integrals that determine the equations of motion. But as has been shown before [3, p. 13] the choice of the coordinate system in $\underset{4}{\gamma_{mn}}$, $\underset{4}{\gamma_{00}}$, $\underset{5}{\gamma_{m0}}$, has no influence upon the equations of motion of the sixth order. But it will have influence upon the equations of motion of the eighth order. Therefore it is possible to use such a coordinate system for $\underset{3}{\gamma_{0m}}$, $\underset{4}{\gamma_{mn}}$, $\underset{4}{\gamma_{00}}$, $\underset{5}{\gamma_{0m}}$, so that the contributions to the surface integrals coming from $\underset{8}{\Lambda}$ are zero. We know [5], that such a coordinate system exists, though it would not be an easy task to find it explicitly. Similarly we can introduce such coordinate conditions in $\underset{6}{\gamma_{mn}}$, $\underset{6}{\gamma_{00}}$, $\underset{7}{\gamma_{m0}}$ that the contributions to the surface integral $\underset{10}{\Lambda}$ will be zero. Thus it is possible to regard the equations of motion that we obtained before [2] as the *exact* equations of motion in a properly chosen coordinate system. We can get rid of all contributions to the surface integral beginning with $\underset{8}{\Lambda}$, but we can not get rid of $\underset{4}{\Lambda}$ and $\underset{6}{\Lambda}$; these expressions, and also the equations of motion up to the sixth order, are determined not by the choice of our coordinate system, but essentially by the procedure concerning the use of our new approximation method.

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Institute for Theoretical Physics and
State Mathematical Institute, Warsaw

MULTIPLE QUANTIZATION

A. E. SCHEIDEGGER

1. Introduction. The efforts of most theoretical physicists of this century have been directed towards that branch of the physical science which is commonly called "Quantum Theory." Physically, Quantum Theory was postulated because of a vast amount of physical evidence which led to the postulates of states, observables, superposition, and commutation relations. From these four postulates, all quantum mechanics follows.

It is well known that the differential equations which are obtained in Quantum Theory can be quantized for a second time. There is a strong physical motive for doing this, namely, the necessity for a field description, required both on general theoretical grounds and to explain, for example, pair production. Mathematically the "second" quantization is expressed in exactly the same way as the first one. In this fashion a certain mathematical scheme has been set up for "quantizing" a differential equation. This scheme consists of several stages whereby the original dynamical variables are finally replaced by operators. This scheme has been applied to many examples. Because of the fact that it is mathematically rather involved, this was not an easy task.

Once the mathematical scheme of Quantum Theory has been obtained, it is of considerable interest to investigate the latter even without any further reference to its physical background. It is possible that the mathematical form suggests interesting generalizations which, in the end, might even have a physical application. This is what we propose to do here.

A careful investigation of the mathematical formalism of Quantum Theory shows that it can be regarded as a process which can be applied to almost any differential equation. Thus one might say that quantization is a corrective mathematical process which is applicable to unsatisfactory physical differential equations. This is an entirely new aspect of quantum mechanics which is quite apart from any physical arguments for introducing it. It is suggested by the mathematical form of the equations and it remains to be seen whether it has any physical applications.

If we accept the interpretation of quantization as a corrective mathematical process, the possibility of repeated or multiple quantization presents itself, as it seems to us, quite naturally. There are, of course, no direct physical reasons for such a repetition of the quantization process, except for the fact that it yields physical results when applied the first two times. From the mathematical standpoint, however, a further repetition seems a most natural generalization. In addition, it might be of interest to know, even for a physicist, what the quantization process does to his equations when repeatedly applied. Perhaps one

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might speculate that each quantization makes the equations describing a physical process more and more accurate.

Thus, we propose to give first a detailed account of the mathematical inducements for the notion of multiple quantization. The idea thus established will be discussed in the rest of the paper.

The results of the present investigations are as follows.

As a first example the Schroedinger-Gordon field is quantized for the third time. To achieve this, it is necessary to obtain a Schroedinger-representation for the Schroedinger-Gordon field, since the customary Heisenberg matrix-representation does not permit a further quantization. Then, the energy eigenvalues of the third quantization of the Schroedinger-Gordon field are obtained; they are found to be $\frac{1}{2} \sum \hbar\omega_k N_k$.

In order to quantize an equation for an infinite number of times, one has to know what the quantization of a Schroedinger equation with an arbitrary Hamiltonian yields (with every equation after a few steps one seems to arrive at a Schroedinger equation). It is found that in this case the $(n + 1)$th quantization describes a Bose ensemble of systems as described by the nth quantization.

In the case where the quantization is performed using anticommutation, instead of commuation rules, one obtains a Fermi, instead of a Bose ensemble. In this instance it turns out that the repetition of the quantization procedure is, physically, simply an easy way of doing the quantum statistics of fields.

2. Inducements for multiple quantization: notation. First, we should like to show how those ideas which we intend to discuss in this paper suggest themselves quite naturally. In order to do so, we have to summarize the principles of quantization in their mathematical form, emphasizing especially their algebraic features. At the same time, this will serve to explain the notation of the present paper. The physical reasons which require the mathematical steps will not be discussed here, since we are only concerned with the mathematical formalism.

In classical theory one starts with a series of real generalized co-ordinates q_k $(k = 1, \ldots, f)$ which describe the state of the mechanical system concerned. The co-ordinates q_k are functions of a parameter t which, in applications, may be identified with the physical time. Dots denote derivatives with respect to this parameter. The equations of motion follow from a variational principle

$$\delta \int L \, dt = 0, \tag{2.1}$$

where $L = L(q_k, \dot{q}_k, t)$ denotes the Lagrangian function of the system. The variational principle is equivalent to the Euler-Lagrangian differential equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0. \tag{2.2}$$

Instead of using this representation one may pass to the canonical formalism

in the well-known manner. The algebraic nature of the latter is seen if one introduces Poisson brackets between arbitrary functions F, G, . . . of q_k and $p_k = \partial L/\partial \dot{q}_k$. Then, the equations of motion may be written as follows

$$\dot{F} = [F, H], \tag{2.3}$$

H denoting the Hamiltonian function of the system.

The Poisson brackets have some remarkable algebraic properties (see, e.g., Dirac's book [1, p. 85-86]), which permit one to reduce any Poisson brackets of arbitrary functions of p_k, q_k to elementary Poisson brackets between p_k and q_k. Thus, if one adds to the algebraic rules of Poisson brackets

$$[p_k, q_l] = -\delta_{kl} \tag{2.4}$$

(δ = Kronecker symbol), then the problem of motion can be solved by *algebra alone.*

This fact makes the quantization of a classical theory *possible.* The dynamical variables are re-interpreted as algebraic entities for which only laws of multiplication need to be defined. If the meaning of "Poisson bracket" is newly fixed in terms of the commutator of the two entities concerned, then the motion of the quantum mechanical variables is entirely determined by the Poisson bracket rules and (2.3).

One can obtain a representation of the quantum mechanical algebraic entities by identifying them with linear Hermitian operators in a Hilbert space. Actually, a proper Hilbert space is not quite sufficient because one has to admit vectors of infinite length. Nevertheless, we shall refer in this paper to such a general space simply by calling it "Hilbert" space. This draws one's attention to the elements of such a generalized Hilbert space. Mathematically, they are a kind of vectors denoted by φ, to each of which a dual exists (φ^*) so that a scalar product $\langle \varphi^* \varphi \rangle$ may be formed. Physically, they are interpreted as representatives of the mechanical system such that

$$\bar{A} = \langle \varphi^* (A\varphi) \rangle \tag{2.5}$$

is the average value for the dynamical observable A in a statistical ensemble of measurements.

The representation of quantum mechanics in this form corresponds to the Heisenberg-representation. By throwing the time-dependence from the operators into the Hilbert vectors φ, one can pass over to the Schroedinger representation. Thus, one arrives at a differential equation for the time-dependence of the Hilbert vector φ:

$$\dot{\varphi} = \frac{i}{\hbar} H \varphi. \tag{2.6}$$

In short, we note that starting from a "vector"

$$\vec{q} = (q_1 \ldots q_k), \tag{2.7}$$

which satisfies a differential equation (2.2), one finally arrives through the process of quantization at a new differential equation of a similar type. The vector $\vec{q}$ originally characterizing the state of the system is replaced by the vector φ serving the same purpose.

Thus, quantization is mathematically a correcting process for "classical" equations. But since the results are equations similar to those from which one had started, it must be asked why this correction should be applied only *once* to the classical theory.

3. General theory. We note that in order to make a quantization procedure according to the canonical rules established in the last section, one needs a Hilbert vector $q(k, t)$ which depends on a time-like parameter. The Hilbert vector must be subject to a differential equation of the type (2.2) regarding its time-dependence:

$$F(q(k\,t), \dot{q}(k, t)\ \bar{q}(k, t)) = 0, \tag{3.1}$$

so that the canonical formalism can be set up. After this has been done, one can proceed to the Heisenberg equations of motion and hence to the Schroedinger equations of motion

$$\frac{\hbar}{i}\,\dot{\psi}(x) = H\psi(x), \tag{3.2}$$

which complete the circle from a Hilbert vector to a Hilbert vector.

If (3.1) is of the type of a Schroedinger equation like (3.2), which will be usually the case if one wants to quantize for the second or third time, there are several difficulties to reckon with. First we note that x in (3.2) is a continuous variable with an infinite range, whereas k in q_k assumes only a finite and discrete set of values. This difficulty, however, has been overcome successfully in the quantum theory of fields.

Furthermore, the ψ's in (3.2) are complex, and the differential equation (3.2) contains only first-order time-derivatives. The latter fact is insignificant, as (3.1) can be replaced by first-order equations; but the reality of the components of the ψ's is essential if one wants to maintain the analogy to the ordinary (first) quantization. We shall see, first, how this reality can be obtained by transforming (3.2), although there exists the possibility of abandoning the correspondence between ψ and a canonical variable, which is indeed sometimes done in the quantum theory of fields.

In order to obtain real equations of the form of (3.1) from (3.2), one has to split ψ into its real and imaginary parts

$$\psi = \psi_1 + i\psi_2 \tag{3.3}$$

where both ψ_1 and ψ_2 are real. Then one has to try to set up equations for ψ_1 and ψ_2 alone. Splitting (3.2) into its real and imaginary parts yields, if H is assumed to be real (which will be ordinarily the case),

$$\hbar\dot{\psi}_1 = -H\psi_2, \qquad \hbar\dot{\psi}_2 = H\psi_1. \tag{3.4}$$

From the first of these equations one obtains

$$\psi_2 = -\hbar H^{-1}\dot{\psi}_1, \tag{3.5}$$

and from the second

$$\psi_1 = \hbar H^{-1}\dot{\psi}_2 \tag{3.6}$$

which, inserted into either equation (3.4), yields

$$-\hbar^2\ddot{\psi}_1 = H H \psi_1 \tag{3.7}$$

and an identical equation for ψ_2. It maybe noted that the Schroedinger-Gordon equation is of this form (3.7).

However, as it was said above, it is not necessary to transform (3.2) into real equations. If one keeps equation (3.2), one has to abandon the direct analogy between ψ and ordinary dynamical variables. Thus it may be possible to derive (3.2) from a Lagrangian principle wherein the Lagrangian function will depend on ψ and ψ^* as well as on their derivatives. Then one can formally define the canonical conjugates π and π^* and write down the Hamiltonian. Doing this it should be kept in mind that π and ψ are no longer canonically conjugate in the *classical* sense; it would be difficult to define Poisson brackets between functionals depending on them. One may, however, assume the same commutation relations as if π and ψ *were* conjugate.

This treatment is very useful if one is only interested in the matrix formulation of the quantized theory. Then, π and ψ are simply represented by non-Hermitian matrices. This is sometimes so convenient that even in the case where one starts with equation (3.7), one transforms the latter into an equation where the variables are no longer real; this is well known from the transition to momentum space in the Schroedinger-Gordon case. Using complex variables, one is, however, generally at a loss if one wants to set up the Schroedinger formulation of the theory in order to proceed with further quantizations. In this case, one has to transform back to variables which are canonical.

In the next section, we shall demonstrate these ideas in a model example.

4. Example: Third quantization[1] of the Schroedinger-Gordon equation. Assuming the Hamiltonian

$$H = c\sqrt{p^2 + m^2c^2} \tag{4.1}$$

leads, after the *first* quantization, to the Schroedinger-Gordon equation (see [6]):

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu^2\right)\psi = 0 \tag{4.2}$$

[1]The name "third quantization" has been used by Nambu [4] in quite a different context. Our paper has nothing to do with the work of Nambu.

which can be written in the following Hamiltonian form:

$$H = \int \mathfrak{H}\, d^3x; \quad \mathfrak{H} = \tfrac{1}{2}\{\pi^2 + c^2|\text{grad}\ \psi|^2 + c^2\mu^2\psi^2\}. \tag{4.3}$$

If π, ψ are taken as Hermitian operators, (4.3) represents the *second* quantization. A Fourier transform of the variables leads to

$$H = \tfrac{1}{2}\sum_k \{p_k^*\, p_k + \omega_k^2\, q_k^*\, q_k\}. \tag{4.4}$$

This equation is usually solved by the matrix method. If we want to carry on our quantization procedure for the *third* time, we must establish first a Schroedinger equation for the second quantization, instead of (4.4). The transition from a Heisenberg to a Schroedinger formulation is usually done in such a way that one of the canonical variables is taken as the argument of the new Schroedinger function and the conjugate one represented as a differential operator. Unfortunately, this procedure cannot be applied to (4.4) because p and q are not canonical since they are not real. They satisfy, however, the following relations (see [6]):

$$p_k^* = p_{-k};\ q_k^* = q_{-k}. \tag{4.5}$$

The ordinary transition from ψ, π to q, p is thus not a canonical transformation. Therefore, one has to introduce for each "dimension" k a variable ξ_k which *is* canonical. If one finds a representation of p_k, q_k by differential operators acting upon the ξ_k's such that the commutation relations and the conditions (4.5) are satisfied, then one has obtained the *Schroedinger representation of the second quantization*. It is easy to verify that this can be done by setting

$$q_k = \tfrac{1}{2}\left(\frac{\hbar}{\omega_k}\right)^{\frac{1}{2}} \left\{\xi_k + \xi_{-k} - \frac{\partial}{\partial \xi_k} + \frac{\partial}{\partial \xi_{-k}}\right\} \tag{4.6a}$$

$$p_k = \tfrac{1}{2}(\hbar\omega_k)^{\frac{1}{2}}\, i\left\{\xi_k - \xi_{-k} + \frac{\partial}{\partial \xi_k} + \frac{\partial}{\partial \xi_{-k}}\right\} \tag{4.6b}$$

A straightforward check shows that the commutation relations for p, q are satisfied as well as (4.5). For the latter condition one has to remember that the star indicates the Hermitian adjoint of an operator and that the Hermitian adjoint of $\partial/\partial\xi$ is $-\partial/\partial\xi$. The procedure used here has been investigated by Infeld and Hull [2, §6] to which the reader is referred for the details. The total Hamiltonian of the field may be represented by

$$H = \sum \tfrac{1}{2}\hbar\omega_k(\xi_k^2 - \partial^2/\partial\xi_k^2). \tag{4.7}$$

Since we can split up the eigenfunctions for every "dimension" k, it is possible to make the following "ansatz":

$$\phi = \prod_k \exp\left(\frac{i}{\hbar} E_k\, t\right)\varphi_k(\xi_k). \tag{4.8}$$

Then, the Schroedinger equation

$$H\,\phi = \frac{\hbar}{i}\dot{\phi} \tag{4.9}$$

yields the following condition for the φ_k's:

$$\tfrac{1}{2}\hbar\omega_k(\xi_k^2 - \partial^2/\partial\xi_k^2)\varphi_k = E_k\,\varphi_k. \tag{4.10}$$

As is well known, equation (4.10) is soluble only for

$$E_k = \tfrac{1}{2}\hbar\omega_k\,(2N_k + 1), \tag{4.11}$$

which is the same result as obtained by the matrix method [2, p. 40]. A general state vector is obtained by a linear superposition of the particular ones in (4.8).

The *third* quantization of the Schroedinger-Gordon equation is performed if we are able to quantize (4.9), or (4.10). If we keep the time dependence in the φ's, (4.10) may be written

$$\tfrac{1}{2}\hbar\omega_k(\xi_k^2 - \partial^2/\partial\xi_k^2)\,\varphi_k = \frac{\hbar}{i}\,\dot{\varphi}_k. \tag{4.12}$$

The task is to quantize (4.12). We note that this equation is of the form of a non-relativistic Schroedinger equation, if we set

$$m = \hbar/\omega_k;\;\; U = \tfrac{1}{2}\xi_k^2\,\hbar\omega_k. \tag{4.13}$$

The Hamiltonian formulation of the non-relativistic Schroedinger field has been set up long ago [5, p. 338]. Using those earlier results, we can write down the Hamiltonian which refers to the "dimension" k:

$$H_k = \tfrac{1}{2}i\omega_k\int\{\text{grad}\,\pi_k\,\text{grad}\,\varphi_k + \xi_k^2\,\pi_k\,\varphi_k\}\,d\xi_k. \tag{4.14}$$

Thus, the *total* Hamiltonian is simply the sum of the expressions (4.14) taken over all the "dimensions" k:

$$H = \tfrac{1}{2}\,i\sum_k \omega_k\int(\text{grad}\,\pi_k\,\text{grad}\,\varphi_k + \xi_k^2\,\pi_k\,\varphi_k)\,d\xi_k. \tag{4.15}$$

The eigenvalues of the total Hamiltonian are of particular interest. We note that the eigenvalues of each H_k have been calculated; they are

$$(H_k)\ldots = \sum N_n^{(k)}E_n^{(k)}, \tag{4.16}$$

where the N_n's range from 0 to ∞, and $E_n^{(k)}$ is eigenvalue of the equation

$$\left(-\tfrac{1}{2}\hbar\omega_k\frac{\partial^2}{\partial\xi_k^2} + \tfrac{1}{2}\hbar\omega_k\,\xi_k^2\right)\Psi_n(\xi_k) = E_n^{(k)}\Psi_n(\xi_k). \tag{4.17}$$

Thus we have

$$E_n^{(k)} = \tfrac{1}{2}\hbar\omega_k(2n + 1). \tag{4.18}$$

Therefore, we find the following eigenvalues of H_k:

$$(H_k)\ldots = \tfrac{1}{2}\hbar\omega_k\sum_n N_n^{(k)}(2n + 1), \tag{4.19}$$

and of H:

$$(H)\ldots = \sum_{k,n} \tfrac{1}{2}\hbar\omega_k N_n^{(k)}(2n+1). \tag{4.20}$$

We may observe that the $N_n^{(k)}$'s are arbitrary integer numbers ranging from 0 to ∞. But then also

$$\mathfrak{N}_k = \sum_n (2n+1)N_n^{(k)} \tag{4.21}$$

are arbitrary integer numbers ranging from 0 to ∞, and we might as well represent the eigenvalues of H in the following from

$$(H)\ldots = \sum \tfrac{1}{2}\hbar\omega_k \mathfrak{N}_k. \tag{4.22}$$

These eigenvalues are very similar to those which were obtained after the second quantization (cf. (4.11)). Speaking in terms of particles we can say that our equations describe an ensemble of $\mathfrak{N}_k$ particles of the energy $\frac{1}{2}\hbar\omega_k$, k ranging over all the lattice points in momentum space.

However, it is interesting to observe that the "particles" have only half the energy of those of the Schroedinger-Gordon equation. Furthermore, the infinite term in the energy has disappeared, since the lowest state is $\mathfrak{N}_k = 0$ for every k. Therefore, it seems better to interpret (4.22) physically in a different way so that the usual physical interpretation of the second quantization can be retained. It is easily seen that the present scheme can be taken as describing a Bose *ensemble of fields* (instead of an *ensemble of particles*) each of which satisfies the Schroedinger-Gordon equation. In this instance the new quantization is equivalent to considering Bose statistics of the meson fields.

It will be seen in the next section that quantizing a Schroedinger equation can always be interpreted as taking a Bose ensemble of systems as present before the new quantization, and doing statistics with it. Thus our last remark is in agreement with more general principles.

5. Infinite quantization. After having seen that it should always be possible to perform a quantization of a first- or second-order differential equation, one may wonder how often this process can be repeated.

We may note that after a very few steps, one always arrives at a Schroedinger type of equation; i.e.,

$$\dot{\phi} = \frac{i}{\hbar}{}^*H\,\phi. \tag{5.1}$$

The arguments of the ϕ-function are the parameter t and the canonical variables *before* the quantization. This is due to the fact that one has to set up the Hamiltonian formalism for the equation from which one wants to start, and the Hamiltonian formalism immediately leads to (5.1). One may check that with the Schroedinger-Gordon case where we arrived at a Schroedinger equation after the *third* step, as was seen in the last section.

Therefore, if we are able to find a recursion formula which connects the eigenvalues of ${}^{n}H$ in (5.1) with the eigenvalues of the Hamiltonian ${}^{n+1}H$ of the next quantization, then we are able to proceed with the quantizations as often as we like.

In order to obtain that (5.1) is the Euler differential equation of a Lagrangian principle, the Lagrangian density must be of the form

$$\mathfrak{L} = i\hbar\,\phi^*\dot{\phi} + B(\phi, \phi^*) \tag{5.2}$$

so that the canonically conjugate π of ϕ is

$$\pi = \frac{\partial \mathfrak{L}}{\partial \dot{\phi}} = i\hbar\,\phi^*. \tag{5.3}$$

It would be rather difficult to find all the terms in the Lagrangian (5.2) explicitly so that such a general equation as (5.1) would follow from it. Nevertheless, we can find the Hamiltonian ${}^{n+1}H$ in matrix representation, in terms of the matrix elements of ${}^{n}H$, provided that we *assume the existence* of such a Hamiltonian formalism and the corresponding quantization.

For, if such a Hamiltonian formalism and the corresponding quantization exists, then the consistency of (5.1) with the quantum formalism requires

$$\dot{\phi} = \frac{i}{\hbar}\,[\phi, {}^{n+1}H] = \frac{i}{\hbar}\,{}^{n}H\,\phi \tag{5.4}$$

or

$$[\phi, {}^{n+1}H] = {}^{n}H\,\phi. \tag{5.5}$$

Expanding ϕ into the complete orthogonal set of eigenfunctions of ${}^{n}H$ yields

$$\phi = \sum a_m\, u_m(x), \tag{5.6}$$

$$\pi = i\hbar\,\phi = i\hbar \sum a_m{}^*\, u_m{}^*(x). \tag{5.7}$$

The canonical commutation relations for π and ϕ are satisfied if

$$[a_m, a_{m'}] = [a_m{}^*, a_{m'}{}^*] = 0;\quad [a_m, a_{m'}{}^*] = \delta_{mm'} \tag{5.8}$$

holds. Inserting the expansions for π and ϕ into (5.5) yields

$$\sum a_m\, u_m(x)\, {}^{n+1}H - {}^{n+1}H \sum a_m\, u_m(x) = \sum {}^{n}E_m\, u_m(x) a_m, \tag{5.9}$$

where ${}^{n}E_m$ denotes the mth eigenvalue of ${}^{n}H$. We multiply (5.9) by $u_k{}^*$ and integrate over x; one then obtains

$$a_k\, {}^{n+1}H - {}^{n+1}H\, a_k = {}^{n}E_k\, a_k, \tag{5.10}$$

which is a matrix equation for ${}^{n+1}H$. The solution is obviously

$$ {}^{n+1}H = \sum_j {}^{n}E_j\, a_j{}^* a_j, \tag{5.11}$$

because of the following relation which holds in virtue of (5.8):

$$(5.12)\qquad [a_k, {}^{n+1}H] = \sum_j {}^nE_j(a_k a_j^* a_j - a_j^* a_j a_k) = {}^nE_k[a_k, a_k^*]a_k = {}^nE_k a_k.$$

A diagonal representation of ${}^{n+1}H$ is obtained if we choose

$$(5.13)\qquad a_k = \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & \ddots & \ddots \end{pmatrix};\quad a_k^* = \begin{pmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & \ddots & \\ & & \ddots & \ddots \end{pmatrix};\quad N_k = a_k^* a_k = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & \ddots \end{pmatrix}$$

which obviously conforms with (5.8). (The matrices (5.13) refer to the index k only, a_k is the unit matrix with respect to all other indices.) Thus, the diagonal elements of ${}^{n+1}H$ become

$$(5.14)\qquad {}^{n+1}E_m = \sum N_k {}^nE_k;\quad m = (N_0 N_1 \ldots).$$

This means that the $(n+1)$th quantization simply describes a statistical ensemble of systems as described by the nth quantization, wherein the counting prescriptions of Einstein and Bose hold. (Each combination of N_k's is enumerated just once.[2]) The lowest eigenvalue is always zero, no matter what the ${}^nE_k$'s are; it never diverges.

If we perform one further quantization, the eigenvalues will be

$$(5.15)\qquad {}^{n+2}E_j = \sum_m N_m \sum N_k' {}^nE_k = \sum \mathfrak{N}_k {}^nE_k$$

where $\mathfrak{N}_k$ again may assume any integer values from 0 to ∞.

Therefore we see that the eigenvalues of the energy do not change any more after one has had a Schroedinger equation and quantized it once more. Thus, even if one repeats quantization an *infinite number of times*, the energy eigenvalues remain final after a few steps. Therefore the eigenvalues of the Schroedinger-Gordon equation will always be as given in (4.22), notwithstanding how many quantizations are performed afterwards.

6. The anticanonical formalism. It seems desirable to find out what becomes of the formalism of the last section in the case of anticanonical quantization.

The anticanonical quantization is distinguished from the canonical procedure by the fact that for π, ϕ anticommutation (rather than commutation) relations are assumed. The equation of motion, however, holds in its canonical form.

In order to find the anticanonical formulae that correspond to what we have done in the last section, one can keep practically all the equations that were developed there, except that one has to find a new representation for the matrices a_k and a_k^*. These representations have been proposed by Jordan and Wigner [3] in connection with the quantization of the Dirac electron field. The same matrices may be used in the *general* formalism as required in the present investigations. Thus we choose

[2] It has been known before that a Schroedinger equation with an arbitrary *potential* leads to a Bose ensemble of the non-quantized system [5]; but the proof had never been given for an arbitrary *Hamiltonian*, as far as I know.

$$a_k = \eta_k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{6.1}$$

$$a_k^* = \eta_k \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.2}$$

The index k labels the state of the system as before. The factors η_k are equal to $+1$ or -1 according to whether the number of occupied states with labels $n < k$ is even or odd.

Using this representation it is easy to see that the Hamiltonian is again given by

$$^{n+1}H = \sum_j {}^nE_j \, a_j^* \, a_j. \tag{6.3}$$

This satisfies indeed the matrix equation (5.10).

This means that the $(n+1)$th anticanonical quantization of a Schroedinger equation describes a statistical ensemble of such systems as are described by the nth Schroedinger equation. But now, since we are using the anticanonical formalism, we see that this is *a Fermi-Dirac ensemble.*

This result holds under the same premises and assumptions as before, that is, the *consistency* of the formalism has to be presumed in each case.

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Imperial Oil Limited
Calgary, Alberta

THE PROPAGATION OF A PLANE SHOCK INTO A QUIET ATMOSPHERE

M. H. MARTIN

Introduction. In this note the problem of determining the propagation of a plane shock moving through a polytropic gas into an undisturbed body of the gas, leaving a non-isentropic disturbance behind it, is reduced to the solution of the problem of Cauchy for a Monge-Ampère partial differential equation of special type. The application of this result to specific examples is being studied with a view to later publication.

1. First principles. In one-dimensional unsteady flow of a gas the velocity u, density ρ, and pressure p must satisfy [3, p. 508] the underdetermined system

$$\rho(u_x u + u_t) + p_x = 0, \quad (\rho u)_x + \rho_t = 0, \tag{1}$$

of partial differential equations implied by the physical hypotheses of conservation of momentum and of mass. By multiplying the second equation by u and adding the result to the first, we arrive at the equivalent system

$$(p + \rho u^2)_x + (\rho u)_t = 0, \quad (\rho u)_x + \rho_t = 0.$$

If u, ρ, p are functions of x, t which satisfy this system, we can infer the existence of two functions $\bar{\xi}$, ψ of x, t defined by

$$d\bar{\xi} = \rho u\, dx - (p + \rho u^2)\, dt, \quad d\psi = \rho\, dx - \rho u\, dt.$$

Along a trajectory $x = x(t)$ of a gas particle in the (x,t)-plane we have $dx/dt = u$ and consequently ψ is constant along such a trajectory. The curves $\psi(x, t)$ is constant in the (x, t)-plane are accordingly the trajectories of gas particles and the function ψ may therefore be termed the *trajectory function*.

One readily sees that the above system can be given the form

$$d\bar{\xi} = -p\, dt + u d\psi, \qquad d\psi = \rho\, dx - \rho u\, dt,$$

which, in turn can be replaced by

$$d\xi = u d\psi + t dp, \qquad d\psi = \rho\, dx - \rho u\, dt, \tag{2}$$

provided we set $\xi = \bar{\xi} + pt$.

The form of this system suggests that ψ, p be taken for independent variables. This amounts to introducing the trajectories and isobars in the (x, t)-plane as a system of curvilinear coordinates in this plane. Obviously this cannot be done if the trajectories and isobars coincide and we accordingly exclude from consideration those flows in which each gas particle retains a constant pressure

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during its motion.[1] The introduction of ψ, p as independent variables into the system (2) yields the relations

$$u = \xi_\psi, \quad t = \xi_p, \quad x_\psi = \xi_\psi\, \xi_{\psi p} + \tau, \quad x_p = \xi_\psi\, \xi_{pp},$$

where $\tau = \rho^{-1} = \tau(\psi, p)$, is the specific volume. When x is eliminated by partial differentiation from the last two equations above we find that $\xi = \xi(\psi, p)$ is a solution of the Monge-Ampère partial differential equation.[2]

$$\xi_{\psi\psi}\, \xi_{pp} - \xi_{\psi p}^{\,2} = \tau_p. \tag{3}$$

Here $\tau = \tau(\psi, p)$ is an arbitrary function whose appearance reflects the underdeterminedness of the system (1). Once $\tau(\psi, p)$ has been specified and a solution $\xi = \xi(\psi, p)$ of this equation has been found an unsteady flow is presented by

$$x = x(\psi,p) = \int \{(\xi_\psi\, \xi_{\psi p} + \tau)d\psi + \xi_\psi\, \xi_{pp}\, dp\}, \quad t = \xi_p(\psi,p), \; u = \xi_\psi(\psi,p). \tag{4}$$

For a given ψ the first two equations present the trajectory of a gas particle in the (x, t)-plane parametrically in terms of the pressure p; the remaining equation provides the velocity u at each point of the trajectory, and the density may be obtained from $\tau = \tau(\psi, p)$.

The arbitrariness in the function $\tau(\psi, p)$ entering into (3) may be limited by further hypotheses of a physical nature.

To begin with, let us suppose that the gas is polytropic with the caloric equation of state [1, p. 10]

$$p = e^{(S-S_*)/c_v} \cdot \tau^{-\gamma},$$

which we write in the form

$$\tau = e^{(S-S_*)/c_v} \cdot p^{-n}, \qquad n = 1/\gamma.$$

Secondly, from the hypothesis of conservation of energy it follows that the specific entropy S is constant along a trajectory [1, pp. 15-16], i.e., $S = S(\psi)$, the function $S(\psi)$ being termed the *entropy distribution function*. Thus the arbitrary function $\tau(\psi, p)$ must take the special form

$$\tau = \delta(\psi)p^{-n}, \qquad n = 1/\gamma, \tag{5}$$

where $\delta(\psi)$ is an arbitary function, determined by the choice of the entropy distribution function.

2. Progressive condensation shock. For a condensation shock, carrying in back of it the values u, τ, p of the velocity, specific volume, and pressure, traveling into a quiet atmosphere in which these quantities have the constant values u_0, τ_0, p_0, the shock conditions take the form [3, pp. 512-513]

[1] Although I have not investigated the point, I suspect that flows with this property form a very restricted class.

[2] This equation was obtained earlier by the author as a minor formal modification of a method developed for the treatment of steady plane flows. See [2, pp. 149–150].

$$
\begin{aligned}
&\text{(i)} \quad u = u_0 + \sqrt{(p - p_0)(\tau_0 - \tau)}, \\
(6) \qquad &\text{(ii)} \quad \hat{x} = u_0 + \tau_0 \sqrt{(p - p_0)/(\tau_0 - \tau)}, \\
&\text{(iii)} \quad \frac{p}{p_0} = \frac{(\gamma + 1)\tau_0 - (\gamma - 1)\tau}{(\gamma + 1)\tau - (\gamma - 1)\tau_0},
\end{aligned}
$$

where $\hat{x}$ denotes the velocity of propagation of the shock with respect to a fixed plane.

We shall prove the following theorem for a polytropic gas.

THEOREM. *Once the entropy distribution function $S(\psi)$ is specified, the determination of a condensation shock moving into a quiet atmosphere (and the flow immediately in back of it) reduces to the solution of a problem of Cauchy for the Monge-Ampère partial differential equation*

$$\xi_{\psi\psi}\, \xi_{pp} - \xi_{\psi p}^{\,2} + n\delta(\psi) p^{-n-1} = 0.$$

If we substitute τ from (5) into (6 iii), we find

$$(7) \qquad \mu^2 \tau_0\, p^{n+1} - \delta(\psi) p + p_0 \tau_0\, p^n - \mu^2 \delta(\psi) p_0 = 0, \qquad \mu^2 = (\gamma - 1)/(\gamma + 1).$$

This equation defines p as a function of ψ and the curve $p = p(\psi)$ in the (ψ, p)-plane is termed the *carrier*.

From (4) and (6 i) the values of ξ_ψ are prescribed along the carrier by

$$\xi_\psi = u_0 + \sqrt{\{\tau_0 - \delta(\psi) p^{-n}(\psi)\}\{p(\psi) - p_0\}}.$$

To obtain the values prescribed for ξ_p along the carrier, we employ (6 ii). First we observe, from (4), that

$$\hat{x} = \xi_\psi + \tau \Big/ \frac{d\xi_p}{d\psi},$$

and consequently, from (6 ii),

$$\frac{d\xi_p}{d\psi} = \sqrt{\{\tau_0 - \delta(\psi) p^{-n}(\psi)\}/\{p(\psi) - p_0\}},$$

to determine ξ_p along the carrier up to an arbitrarily additive constant. Of the three shock conditions (6), the last serves to determine the carrier in the (ψ, p)-plane, and the first two provide the required Cauchy data upon the carrier.

If $\xi = \xi(\psi, p)$ is a solution of this problem of Cauchy, the shock curve in the (x, t)-plane and the flow behind it is obtained, at least locally, by mapping one side of the neighbourhood of the carrier in the (ψ, p)-plane on to the (x, t)-plane by the first two formulae in (4).

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Institute for Fluid Dynamics and Applied Mathematics
College Park, Maryland

CORRESPONDING POLYHEDRA IN THE THREE SPACES OF CONSTANT CURVATURE

ERNST ROESER

1. Introduction. The five Platonic solids can be drawn in elliptic or hyperbolic space just as well as in Euclidean space. Their numerical properties are, of course, the same in all three. So are the various angles subtended at the centre. But the face-angles and dihedral angles are greater in elliptic space, smaller in hyperbolic. It is a special feature of the non-Euclidean spaces that we cannot change the size of a solid without changing its shape. The edge-length $2l$, circum-radius R, and in-radius r are conveniently expressed as functions of the *mid-radius* ρ: the distance from the centre to the mid-point of an edge. With each non-Euclidean polyhedron we shall associate a corresponding Euclidean polyhedron of a definite size, whose mid-radius serves as a parameter for expressing all the metrical properties of the non-Euclidean polyhedron. When an elliptic polyhedron and a hyperbolic polyhedron have the same parameter, the edge-length of the former is the complement of the angle of parallelism corresponding to the edge-length of the latter.

For these purposes it is immaterial whether we take the space of positive curvature to be elliptic (single-elliptic) or spherical (double-elliptic). To simplify the formulae we shall employ the natural unit of length. In hyperbolic space this means that a horocyclic arc of length 1 has the tangent at either end parallel to the diameter through the other end.

2. Corresponding polygons. Let us begin by recapitulating the results of an earlier paper [5].

In Euclidean space, spherical triangles are usually drawn on a sphere of radius 1, because then the geodesic lengths of their sides are equal to the angles subtended at the centre of the sphere. In hyperbolic space the same result is achieved by using a sphere of radius

$$\operatorname{arg\,sinh} 1 = \log (\sqrt{2} + 1).$$

On such a sphere, consider a circle of angular radius $\sigma_s \leqslant \frac{1}{2}\pi$ or straight radius σ_h, so that

$$\sinh \sigma_h = \sin \sigma_s \leqslant 1. \tag{1}$$

Through this circle draw a plane and a horosphere (say, for definiteness, the horosphere curved in the same direction as the sphere). Draw p tangents to the circle in its plane, so as to form a regular p-gon, $\{p\}$, of in-radius σ_h and side

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$2l_h$, say. With the same points of contact, draw p great-circular tangents on the sphere, and p horocyclic tangents on the horosphere, so as to form a spherical $\{p\}$ of in-radius σ_s and side $2l_s$, and a horospherical $\{p\}$ of in-radius σ_e and side

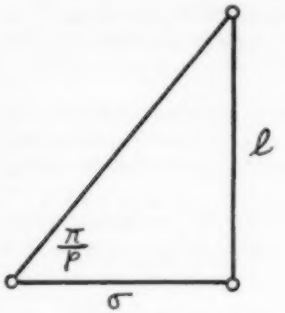

FIG. 1

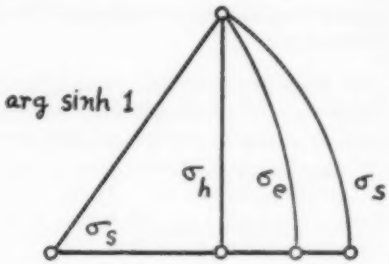

FIG. 2

$2l_e$. The relation between each in-radius and side can be computed from a right-angled triangle (Fig. 1). Using the appropriate trigonometry (hyperbolic, spherical, or Euclidean), we find

$$(2) \qquad \tanh l_h = \sinh \sigma_h \tan \frac{\pi}{p}, \qquad \tan l_s = \sin \sigma_s \tan \frac{\pi}{p}, \qquad l_e = \sigma_e \tan \frac{\pi}{p}.$$

Since σ_h is the semi-chord of a circular arc σ_s and of a horocyclic arc σ_e (Fig. 2), we have

$$(3) \qquad \sinh \sigma_h = \sin \sigma_s = \sigma_e$$

[7, p. 62, with $S = 1$], and therefore

$$(4) \qquad \tanh l_h = \tan l_s = l_e.$$

This last result can also be seen directly, since l_s and l_h are a circular arc and its tangent in a diametral plane of the sphere, while l_e and l_h are a horocyclic arc and its tangent in a diametral plane of the horosphere. In particular, when the hyperbolic $\{p\}$ is "asymptotic" (with its vertices at infinity),

$$l_h = \infty, \qquad l_e = 1.$$

By (1) and (2), $\tanh l_h < \tan \pi/p$; therefore a hyperbolic $\{p\}$ has no corresponding spherical $\{p\}$ if

$$\tanh l_h > \tan \frac{\pi}{p}.$$

(This failure cannot occur when $p = 3$ or 4, but may occur for a pentagon or higher polygon.) By (4),

$$\tan l_s = \tanh l_h < 1;$$

therefore a spherical $\{p\}$ has no corresponding hyperbolic $\{p\}$ if $l_s > \frac{1}{4}\pi$, that is, if the $\{p\}$ is a spherical triangle whose sides are obtuse. Thus the only regular polygon for which the correspondence holds both ways, for every possible length of side, is the *square*.

The equation $\tanh l_h = \tan l_s$ may be expressed geometrically [**4**, p. 3] by saying that $2l_s$ is the complement of Lobatschefsky's angle of parallelism for the distance $2l_h$.

3. Regular polyhedra. Analogous considerations in hyperbolic 4-space yield a definition for "corresponding" polyhedra. Instead of a common in-circle, these have a common *mid-sphere* (sphere touching all the edges) with its radius measured three different ways.

Let $\{p, q\}$, where $(p-2)(q-2) < 4$, denote a regular polyhedron whose faces are p-gons, q at each vertex, in Euclidean or non-Euclidean space, so that the number of edges is [**2**, p. 13]

$$E = 2pq/\,[4 - (p-2)(q-2)].$$

Let ϕ and ψ denote the angles subtended at the centre by a half-edge and by the in-radius of a face. Then

$$\begin{aligned} &\cos\phi = \frac{\cos \pi/p}{\sin \pi/q}, \qquad \sin\phi = \frac{\sin \pi/h}{\sin \pi/q}, \qquad \tan\phi = \frac{\sin \pi/h}{\cos \pi/p}, \\ &\cos\psi = \frac{\cos \pi/q}{\sin \pi/p}, \qquad \sin\psi = \frac{\sin \pi/h}{\sin \pi/p}, \qquad \tan\psi = \frac{\sin \pi/h}{\cos \pi/q}, \end{aligned} \tag{5}$$

where $h = \sqrt{(4E+1)} - 1$ [**2**, pp. 19, 21].

Let R denote the circum-radius, ρ the mid-radius, r the in-radius, σ the in-radius of a face, 2α the dihedral angle, and 2θ the face-angle. These can be

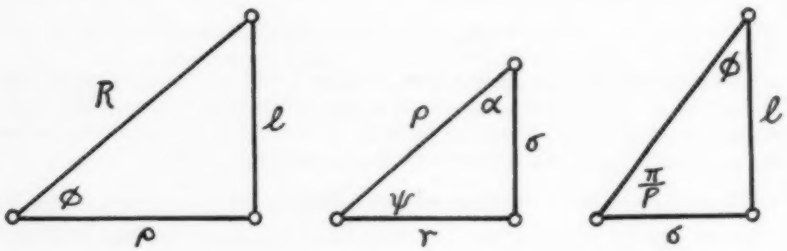

FIG. 3

computed by solving the three right-angled triangles shown in Figure 3. The results are collected in Table I.

The ρ and r of [**4**] have been interchanged for the sake of agreement with Sommerville [**7**, p. 123]. Note, however, that his

are our

a,	α,	θ,	δ,	n,	p,	r_a
$2l$,	2ϕ,	2θ,	2α,	p,	q,	σ.

For the extension to n dimensions see [8, p. 22].

In terms of Schläfli functions [1, pp. 15, 23, 28], the volume of the spherical $\{p, q\}$ is $ES(\frac{1}{2}\pi - \pi/p, \pi/q, \frac{1}{2}\pi - \alpha)$, while that of the hyperbolic $\{p, q\}$ is $EiS(\frac{1}{2}\pi - \pi/p, \pi/q, \frac{1}{2}\pi - \alpha)$.

TABLE I

THE GENERAL REGULAR POLYHEDRON

Elliptic or spherical space	Euclidean space	Hyperbolic space
$\tan l = \sin \rho \tan \phi$	$l = \rho \tan \phi$	$\tanh l = \sinh \rho \tan \phi$
$\tan R = \tan \rho \sec \phi$	$R = \rho \sec \phi$	$\tanh R = \tanh \rho \sec \phi$
$\tan r = \tan \rho \cos \psi$	$r = \rho \cos \psi$	$\tanh r = \tanh \rho \cos \psi$
$\sin \sigma = \sin \rho \sin \psi$	$\sigma = \rho \sin \psi$	$\sinh \sigma = \sinh \rho \sin \psi$
$\cot \alpha = \cos \rho \tan \psi$	$\alpha = \frac{1}{2}\pi - \psi$	$\cot \alpha = \cosh \rho \tan \psi$
$\sin \theta = \sec l \cos \pi/p$	$\theta = \frac{1}{2}\pi - \pi/p$	$\sin \theta = \operatorname{sech} l \cos \pi/p$

4. A digression on rectangular polyhedra. It is interesting to see which non-Euclidean polyhedra $\{p, q\}$ share with the Euclidean cube $\{4, 3\}$ the property that the edges at each vertex form an orthogonal trihedron. This requires

$$q = 3, \qquad \alpha = \theta = \tfrac{1}{4}\pi.$$

Since θ is greater or less than $\frac{1}{2}\pi - \pi/p$ according as the space is elliptic or hyperbolic, we have $p < 4$ in the former case and $p > 4$ in the latter. Thus we find an elliptic rectangular $\{3, 3\}$ with

$$\cos \rho = \cot \psi = \sqrt{\tfrac{1}{2}},$$

a hyperbolic rectangular $\{5, 3\}$ with

$$\cosh \rho = \cot \psi = \tau, \qquad \tau = \tfrac{1}{2}(\sqrt{5} + 1),$$

and also, if we allow ρ to be infinite, a hyperbolic rectangular $\{6, 3\}$ inscribed in a horosphere. In each case the edge $2l$ is given by

$$\cos l \quad \text{or} \quad \cosh l = \sqrt{2} \cos \pi/p.$$

Table II lists the chief properties of these polyhedra, with the cube of mid-radius 1 for comparison. (Note that the mid-radius of the rectangular dodecahedron is equal to its edge: $\rho = 2l$.)

By repeatedly reflecting such a solid $\{p, 3\}$ in its faces, we obtain a regular honeycomb $\{p, 3, 4\}$, filling the whole space.

$\{3, 3, 4\}$, when drawn on a hypersphere in Euclidean 4-space, has the same

vertices as the regular cross polytope β_4, analogous to the octahedron β_3 [2, p. 121]. There are sixteen cells $\{3, 3\}$. By identifying antipodes we derive a honeycomb of eight rectangular tetrahedra filling elliptic space [7, p. 124].

$\{4, 3, 4\}$ is the ordinary space-filling of cubes.

$\{5, 3, 4\}$ was described by Schlegel and Sommerville [6, p. 444; 8, p. 17].

$\{6, 3, 4\}$ was described by Coxeter and Whitrow [3, pp. 426, 427].

TABLE II

THE RECTANGULAR POLYHEDRA

Elliptic or spherical $\{3,3\}$	Euclidean $\{4,3\}$	Hyperbolic $\{5,3\}$	Hyperbolic $\{6,3\}$
$R = \frac{1}{3}\pi$, $\sin^2 R = \frac{3}{4}$	$R^2 = \frac{3}{2}$	$\sinh^2 R = \frac{3}{2}\tau$	$R = \infty$
$\rho = \frac{1}{4}\pi$, $\sin^2 \rho = \frac{1}{2}$	$\rho^2 = 1$	$\sinh^2 \rho = \tau$	$\rho = \infty$
$r = \frac{1}{6}\pi$, $\sin^2 r = \frac{1}{4}$	$r^2 = \frac{1}{2}$	$\sinh^2 r = \frac{1}{2}\tau$	$r = \infty$
$l = \frac{1}{4}\pi$, $\cos 2l = 0$	$2l = \sqrt{2}$	$\cosh 2l = \tau$	$\cosh 2l = 2$

5. Corresponding polyhedra. The formulae in Table I apply to polyhedra $\{p, q\}$ in the three spaces separately. In each case we can assign a value to ρ and deduce all the other properties. *Corresponding* polyhedra are given by identifying the Euclidean ρ with the elliptic sin ρ and the hyperbolic sinh ρ, say

$$\sin \rho_s = \rho_e = \sinh \rho_h,$$

whence

$$\tan l_s = l_e = \tanh l_h, \qquad 2l_s = \tfrac{1}{2}\pi - \Pi(2l_h),$$

and

$$\sin \sigma_s = \sigma_e = \sinh \sigma_h.$$

The remaining relations are not quite so simple, but we can express all the properties in terms of ρ_e and l_e ($= \rho_e \tan \phi$), as follows:

$$\tan R_s = \frac{\rho_e}{\sqrt{(1-\rho_e^2)}} \sec \phi, \quad R_e = \rho_e \sec \phi, \quad \tanh R_h = \frac{\rho_e}{\sqrt{(1+\rho_e^2)}} \sec \phi;$$

$$\tan r_s = \frac{\rho_e}{\sqrt{(1-\rho_e^2)}} \cos \psi, \quad r_e = \rho_e \cos \psi, \quad \tanh r_h = \frac{\rho_e}{\sqrt{(1+\rho_e^2)}} \cos \psi;$$

$$\cot \alpha_s = \sqrt{1-\rho_e^2} \tan \psi, \quad \alpha_e = \tfrac{1}{2}\pi - \psi, \quad \cot \alpha_h = \sqrt{1+\rho_e^2} \tan \psi;$$

$$\sin \theta_s = \sqrt{1+l_e^2} \cos \frac{\pi}{p}, \quad \theta_e = \tfrac{1}{2}\pi - \frac{\pi}{p}, \quad \sin \theta_h = \sqrt{1-l_e^2} \cos \frac{\pi}{p}.$$

It is interesting to observe that

$$\cot^2 \alpha_s + \cot^2 \alpha_h = 2 \tan^2 \psi = 2 \cot^2 \alpha_e$$

and

$$\sin^2 \theta_s + \sin^2 \theta_h = 2 \cos^2 \frac{\pi}{p} = 2 \sin^2 \theta_e,$$

whence

$$\cos 2\theta_s + \cos 2\theta_h = 2 \cos 2\theta_e.$$

6. Limitations of the correspondence. In elliptic or spherical space, the largest possible $\{p, q\}$ has

$$R_s = \rho_s = r_s = \tfrac{1}{2}\pi, \qquad l_s = \phi, \qquad \sigma_s = \psi, \qquad \alpha_s = \tfrac{1}{2}\pi, \qquad \theta_s = \pi/q.$$

Referring to a table of the five Platonic solids [2, p. 293], we see that $\phi > \frac{1}{4}\pi$ for the tetrahedron $\{3, 3\}$, $\phi = \frac{1}{4}\pi$ for the octahedron $\{3, 4\}$, and $\phi < \frac{1}{4}\pi$ for the remaining three. Since

$$\tan l_s = \tanh l_h < 1,$$

an elliptic $\{p, q\}$ has no corresponding hyperbolic $\{p, q\}$ if $l_s > \frac{1}{4}\pi$, that is, if the elliptic $\{p, q\}$ is a tetrahedron of edge greater than $\frac{1}{2}\pi$.

In hyperbolic space, the largest possible $\{p, q\}$ has

$$R_h = l_h = \infty, \quad \tanh \rho_h = \cos \phi, \quad \sinh \rho_h = \cot \phi, \quad \theta_h = 0,$$

$$\tanh r_h = \cos \phi \cos \psi = \cot \frac{\pi}{p} \cot \frac{\pi}{q},$$

$$\sinh \sigma_h = \cot \phi \sin \psi = \cot \frac{\pi}{p},$$

$$\cot \alpha_h = \csc \phi \tan \psi = \tan \frac{\pi}{q},$$

[1, p. 28]. Since $\tanh l_h = \sinh \rho_h \tan \phi = \sin \rho_s \tan \phi < \tan \phi$, a hyperbolic $\{p, q\}$ has no corresponding elliptic $\{p, q\}$ if

$$\tanh l_h > \tan \phi$$

(which implies $\phi < \frac{1}{4}\pi$). Thus, for the correspondence to hold, a hyperbolic hexahedron or dodecahedron or icosahedron must not be too large.

The only regular polyhedron for which the correspondence holds both ways, for every possible size, is the *octahedron*.

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Mainz, Germany

A SIMPLE BOUNDING FORMULA FOR INTEGRALS

J. L. SYNGE

1. Introduction and summary. In this paper I establish the following bounding formula for the integral of a function of n variables:

$$\int_D F\,dD = \frac{\Delta}{n+1}\sum_i m_i F_i + E_1 + E_2, \tag{1.1}$$

the "error terms" E_1 and E_2 being bounded by

$$|E_1| \leqslant \bar{F}(D-T), \quad |E_2| \leqslant kMT. \tag{1.2}$$

Here F is the function, D the domain of integration in Euclidean n-space (and also the n-volume of that domain), and $\bar{F}$ an upper bound of $|F|$ in D. D is divided up into congruent cells with $n+1$ vertices (segments on a line, triangles in a plane, tetrahedra in space), the total n-volume occupied by these cells being T, so that $D-T$ is the n-volume of that part of D not covered by the cells. Δ is the n-volume of a cell. The summation runs over all the vertices of the cells, F_i being the value of F at the vertex i, and m_i the number of cells meeting at this vertex. M is an upper bound to $|d^2F/ds^2|$, for differentiation in all directions and at all positions in D, ds being an element of length in the direction of differentiation. The quantity k is a parameter with the dimensions [length]^2, depending only on the shape and size of the cell, and given by

$$k = \tfrac{1}{2}(r_0^2 - P/\Delta), \qquad P = \int_\Delta r^2\,dD, \tag{1.3}$$

where r_0 is the radius of the circumscribed sphere of a cell and P the polar moment of inertia of a cell relative to its circumcentre. In particular

$$\left\{\begin{array}{ll} \text{for } n=1, & k = \tfrac{1}{12}h^2 \qquad (h = \text{length of segment-cell});\\ \text{for } n=2, & k = \tfrac{1}{24}(s_1^2+s_2^2+s_3^2) \quad (s_1, s_2, s_3 = \text{sides of triangular cell});\\ \text{for } n=3, & k = \tfrac{2}{5}(r_0^2 - R^2)\\ & (r_0 = \text{radius of circumscribed sphere},\\ & \ R = \text{distance between circumcentre and centroid of the cell}). \end{array}\right. \tag{1.4}$$

For $n=1$ we would naturally divide D into equal segments, these segments being the cells. Then $D=T$, $E_1=0$, and (1.1), (1.2) may be written

$$\int_a^b F\,dx = \tfrac{1}{2}h(F_0 + 2F_1 + \ldots + 2F_{r-1} + F_r) + E_2, \tag{1.5}$$

$$|E_2| \leqslant \tfrac{1}{12}h^2 M(b-a);$$

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here h is the length of a segment and $F_0, F_1, \ldots$ the values of F at the end points and points of division. This simple formula is very easy to prove, being in fact a by-product in the proof of the Euler-Maclaurin formula [3, p. 142 (6)]. Its significance here is as the prototype of the general multi-dimensional formula (1.1).

If we like we can write (1.1) in a more general form applicable to the case where the cells are not all the same. Or we can improve the bounds by using different upper bounds M in different parts of D. But these considerations complicate the formula and we shall confine ourselves to establishing (1.1).

Geometrical intuition is a great help and the plan we shall adopt is to prove (1.1) for $n = 2$ and then recognize that the same method is available for $n = 3$ and indeed for all values of n.

2. Bounds for a function and its integral when the function vanishes at the vertices of a triangle. Generalization to n dimensions. As a first step towards the proof of (1.1), we consider a triangle ABC and a function ϕ, continuous and with continuous first and second derivatives. We impose on ϕ the condition

$$\phi = 0 \quad \text{at } A, B \text{ and } C, \tag{2.1}$$

and we write as bound for the second derivative

$$|d^2\phi/ds^2| \leqslant M, \tag{2.2}$$

this inequality holding for differentiation in all directions and at all positions in the triangle.

Let G be the circumcentre of ABC and r_0 the radius of its circumscribed circle. Consider the two functions

$$\Phi_1 = \tfrac{1}{2} M(r_0^2 - r^2), \qquad \Phi_2 = \tfrac{1}{2} M(r^2 - r_0^2), \tag{2.3}$$

where r is distance from G. Note that in ABC, Φ_1 is positive and Φ_2 negative, both functions vanishing at A, B, and C. If we draw three-dimensional graphs of these two functions, they enclose a paraboloidal lens, with A, B, and C lying on its edge. We note that for all directions and positions

$$d^2\Phi_1/ds^2 = -M, \qquad d^2\Phi_2/ds^2 = M. \tag{2.4}$$

The essential theorem we require is this: *By virtue of* (2.1) *and* (2.2), *the three-dimensional graph of ϕ within the triangle ABC cannot pass outside the paraboloidal lens formed from Φ_1 and Φ_2; equivalently,*

$$\Phi_2 \leqslant \phi \leqslant \Phi_1 \tag{2.5}$$

in the triangle ABC.

To prove this, consider first the side AB. On it

$$\frac{d^2}{ds^2}(\Phi_1 - \phi) \leqslant 0, \qquad \Phi_1 - \phi = 0 \text{ at } A \text{ and } B. \tag{2.6}$$

Hence $\Phi_1 - \phi$ is positive (or zero) on AB. Similarly $\phi - \Phi_2$ is positive (or zero). Thus on AB, and similarly on BC and CA, the inequalities (2.5) hold.

Now take any point E on AB and join it to C. On EC we have the inequality (2.6) and the boundary conditions

$$\Phi_1 - \phi = 0 \text{ at } C, \qquad \Phi_1 - \phi > 0 \text{ at } E. \tag{2.7}$$

Hence $\Phi_1 - \phi$ is positive (or zero) on CE. Similarly $\phi - \Phi_2$ is positive (or zero). Since lines such as CE traverse the whole triangle, the inequalities (2.5) are established.

It follows then that, for integration over the triangle,

$$\left| \int_\Delta \phi \, dD \right| < \int_\Delta \Phi_1 \, dD = \tfrac{1}{2} M(r_0^2 \Delta - P), \tag{2.8}$$

$$P = \int_\Delta r^2 \, dD,$$

where dD is an element of area, Δ the area of ABC and P the polar moment of inertia with respect to the circumcentre. We may also write this as

$$\left| \int_\Delta \phi \, dD \right| < kM\Delta, \qquad k = \tfrac{1}{2}(r_0^2 - P/\Delta). \tag{2.9}$$

The extension of the above reasoning to three dimensions is very easy. We now take a tetrahedron $ABCD$, with $\phi = 0$ at the vertices and the bounding condition (2.2) as before. The functions Φ_1 and Φ_2 are defined as in (2.3), G being now the centre of the circumscribed sphere and r_0 its radius. The inequalities (2.5) are established in the same manner as above, and we get again the formula (2.9), wherein dD is now an element of volume and Δ the volume of the tetrahedron.

We can carry the same type of reasoning on into space of higher dimensionality or back to $n = 1$. In fact, (2.9) holds for $n = 1, 2, 3, \ldots$.

The factor k in (2.9) agrees with (1.3), and the special values shown in (1.4) are easily obtained from the following equimomental properties [**2**, pp. 23, 27]:

(i) A straight segment of mass m is equimomental to particles each of mass $m/6$ at its ends and a particle of mass $2m/3$ at its centre.

(ii) A triangle of mass m is equimomental to three particles each of mass $m/3$ at the middle points of its sides.

(iii) A tetrahedron of mass m is equimomental to four particles each of mass $m/20$ at its vertices and a particle of mass $4m/5$ at its centroid.

3. Proof of the bounding formulae (1.1), (1.2). Consider a domain D in the plane and a function F in D with the bounds

$$|F| < \bar{F}, \qquad |d^2F/ds^2| < M. \tag{3.1}$$

Divide up D into triangles covering an area T, and so leaving an area $D - T$ between the triangulation and the boundary of D. Then

$$\int_D F\,dD = \int_T F\,dD + E_1, \tag{3.2}$$

where

$$|E_1| < \bar{F}\,(D - T). \tag{3.3}$$

Now define a function F' throughout T by the conditions that $F' = F$ at every vertex and F' is a linear function of the coordinates in each triangle. Thus, in the triangle (x_1, y_1), (x_2, y_2), (x_3, y_3), F' is given by the determinantal equation

$$\begin{vmatrix} F' & x & y & 1 \\ F_1 & x_1 & y_1 & 1 \\ F_2 & x_2 & y_2 & 1 \\ F_3 & x_3 & y_3 & 1 \end{vmatrix} = 0, \tag{3.4}$$

where F_1, F_2, F_3 are the values of F at the vertices.

Write $\phi = F - F'$. Then $\phi = 0$ at the vertices, and

$$d^2\phi/ds^2 = d^2F/ds^2, \tag{3.5}$$

since F' is linear. Hence, by (3.1),

$$|d^2\phi/ds^2| < M. \tag{3.6}$$

Application of (2.9) gives, for integration over a triangle,

$$\left| \int_\Delta \phi dD \right| < kM\Delta, \tag{3.7}$$

where Δ is the area of the triangle and k as in (2.9).

So far we have not assumed the triangles to be congruent with one another. Let us now make this assumption, so that k and Δ are the same for all triangles. Then, for integration over the whole triangulation, we have from (3.7)

$$\left| \int_T \phi\,dD \right| < kMT. \tag{3.8}$$

Now

$$\int_T F\,dD = \int_T F'\,dD + \int_T \phi\,dD, \tag{3.9}$$

and so, by (3.2) and (3.8),

$$\int_D F\,dD = \int_T F'\,dD + E_1 + E_2, \tag{3.10}$$

where E_1 is bounded as in (3.3) and

$$|E_2| < kMT. \tag{3.11}$$

To complete the derivation of (1.1) from (3.10), we have to evaluate $\int_T F'\,dD$. To do this, we integrate (3.4) over the triangle, obtaining

$$\text{(3.12)}\qquad \begin{vmatrix} \int_\Delta F'\,dD & \bar{x}\Delta & \bar{y}\Delta & \Delta \\ F_1 & x_1 & y_1 & 1 \\ F_2 & x_2 & y_2 & 1 \\ F_3 & x_3 & y_3 & 1 \end{vmatrix} = 0,$$

where $(\bar{x}, \bar{y})$ is the centroid of the triangle. This gives at once

$$\text{(3.13)}\qquad \int_\Delta F'\,dD = \tfrac{1}{3}\Delta(F_1 + F_2 + F_3).$$

Summing for all the triangles, we get

$$\text{(3.14)}\qquad \int_T F'\,dD = \tfrac{1}{3}\Delta \sum_i m_i F_i,$$

where i runs over all the vertices and m_i is the number of triangles meeting at the vertex i. When we substitute in (3.10), we get

$$\text{(3.15)}\qquad \int_D F\,dD = \tfrac{1}{3}\Delta \sum_i m_i F_i + E_1 + E_2,$$

which is (1.1) for $n = 2$.

For $n \neq 2$, the initial factor must be changed from $1/3$ to $1/(n+1)$, because in using the analogue of (3.12) in n dimensions the centroid of the cell has coordinates $\bar{x}, \bar{y}, \ldots$, where

$$\text{(3.16)}\qquad \bar{x} = \frac{1}{n+1}(x_1 + x_2 + \ldots + x_{n+1}).$$

But otherwise the reasoning is the same, and so we may regard the bounding formulae (1.1), (1.2) as established for all values of n.

4. Two special triangulations. In dealing with an integral over a plane domain, two triangulations are particularly simple: (i) equilateral triangles, (ii) isosceles right-angled triangles. For them we have:

(4.1) Equilateral triangle of side $2a$: $\Delta = 3^{\frac{1}{2}}a^2$, $k = \frac{1}{2}a^2$.

(4.2) Isosceles right-angled triangle with hypotenuse $2a$: $\Delta = a^2$, $k = \frac{1}{3}a^2$.

If we use N triangles (all of one type or the other), (1.1) gives

(4.3) Equilateral triangles:

$$\int_D F\,dD = 3^{-\frac{1}{2}}a^2 \sum_i m_i F_i + E_1 + E_2,$$

$$|E_2| < \tfrac{1}{2}\,3^{\frac{1}{2}}\,a^4\,NM.$$

(4.4) Isosceles right-angled triangles:

$$\int_D F\,dD = \tfrac{1}{3}a^2 \sum_i m_i F_i + E_1 + E_2,$$
$$|E_2| < \tfrac{1}{3}a^4 NM.$$

At each vertex inside the triangulation, $m_i = 6$ in (4.3) and $m_i = 4$ or 8 in (4.4). On the boundary these values are reduced. If the domain D is such that it can be broken up into equal equilateral triangles or into equal isosceles right-angled triangles, then for such triangulations we have $E_1 = 0$.

5. The method of von Mises. Professor W. Prager has drawn my attention to a paper in which von Mises [1] develops a powerful method of approximating an integral over a plane domain by a weighted sum of the values of the integrand at selected stations. The error term depends on derivatives $d^\mu F/ds^\mu$ and by taking sufficient stations we can make μ as large as we like. Thus the method of von Mises is more elastic than that of the present paper, for which only $\mu = 2$ occurs. Moreover his method does not use triangulation.

Let us compare the results given by the two methods in the simple case where the domain of integration is a triangle with sides s_1, s_2, s_3. Then (1.1), (1.2), (1.4) of the present paper give for the approximation and the error

$$\int_D F\,dD = \tfrac{1}{3}D(F_1 + F_2 + F_3) + E_2, \tag{5.1}$$
$$|E_2| < \tfrac{1}{24} M D(s_1^2 + s_2^2 + s_3^2).$$

To apply the method of von Mises, let us take the stations at the vertices of the triangle. By formula (3) of his paper, weighting factors A_1, A_2, A_3 are to be found to satisfy

$$\begin{aligned} A_1 \quad + A_2 \quad + A_3 \quad &= D, \\ A_1 x_1 + A_2 x_2 + A_3 x_3 &= D\bar{x}, \\ A_1 y_1 + A_2 y_2 + A_3 y_3 &= D\bar{y}, \end{aligned} \tag{5.2}$$

where $(x_1, y_1), \ldots$ are the vertices and $(\bar{x}, \bar{y})$ the centroid. The solution is

$$A_1 = A_2 = A_3 = \tfrac{1}{3}D, \tag{5.3}$$

and formulae (I), (II) of his paper give for the approximation and the error

$$\int_D F\,dD = \tfrac{1}{3}D(F_1 + F_2 + F_3) + E_2', \tag{5.4}$$
$$|E_2'| < \tfrac{1}{3} M [P_0 + \tfrac{1}{3} D(r_1^2 + r_2^2 + r_3^2)],$$

where P_0 is the polar moment of inertia of the triangle with respect to the origin O and r_1, r_2, r_3 the distances of the vertices from O. This bound for the error is minimized by choosing O at the centroid, and then we have

$$|E_2'| < \tfrac{1}{24} M D(s_1^2 + s_2^2 + s_3^2) + MP_c, \tag{5.5}$$

where P_c is the polar moment of inertia with respect to the centroid.

Comparison with (5.1) shows that the method of the present paper gives closer bounds than the method of von Mises. But of course it must be remembered that this particular example favours the method of the present paper, and that in applying the method of von Mises the stations need not be taken at the vertices, nor need there be only three of them.

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The Dublin Institute for Advanced Studies

CONSTANT HOLOMORPHIC CURVATURE

N. S. HAWLEY

We shall present in this paper a certain theorem concerning complex manifolds provided with an Hermitian metric satisfying the Kaehler restriction. The variables $z_1, z_2, \ldots, z_n$ denote local complex coordinates in the manifold and $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$ their conjugates. The subscripts $a, b, c, \ldots$ run from 1 to n and by $\bar{a}$ we mean $a + n \pmod{2n}$, e.g., $\bar{\bar{a}} = a$.

An Hermitian line element is given by

$$ds^2 = g_{a\bar{b}} dz_a d\bar{z}_b \tag{1}$$

(summation over a, b from 1 to n); the coefficients $g_{a\bar{b}}$ shall satisfy the conditions

$$g_{ab} = g_{\bar{a}\bar{b}} = 0, \quad g_{a\bar{b}} = g_{\bar{b}a} = \bar{g}_{\bar{a}b}. \tag{2}$$

In matric notation (2) has the following form:

$$G = \begin{pmatrix} g_{1,n+1}, \ldots, g_{1,2n} \\ g_{n,n+1}, \ldots, g_{n,2n} \end{pmatrix}.$$

If we let $\bar{G}'$ be the transposed conjugate of G, then by (2), $\bar{G}' = G$ and $\mathfrak{G}$, the matrix of the fundamental tensor, is as follows:

$$\mathfrak{G} = \begin{pmatrix} O & G \\ \bar{G} & O \end{pmatrix},$$

so we see that

$$\mathfrak{G}' = \begin{pmatrix} O & \bar{G}' \\ G' & O \end{pmatrix} = \begin{pmatrix} O & G \\ \bar{G} & O \end{pmatrix} = \mathfrak{G}.$$

The line element (1) is said to satisfy the Kaehler condition if

$$\frac{\partial g_{a\bar{b}}}{\partial z_c} = \frac{\partial g_{c\bar{b}}}{\partial z_a}. \tag{3}$$

The relations (2) and (3) thus imply

$$\frac{\partial g_{\bar{a}b}}{\partial \bar{z}_c} = \frac{\partial g_{\bar{c}b}}{\partial \bar{z}_a}. \tag{4}$$

We may form an affine connection and a Riemann-Christoffel tensor from the fundamental tensor by the usual formulae. The only components different from zero of the Riemann-Christoffel tensor are

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(5) $$R_{a\bar{b}c\bar{d}},\ R_{a\bar{b}\bar{c}d},\ R_{\bar{a}bc\bar{d}},\ R_{\bar{a}b\bar{c}d}.$$

A two-dimensional element at a point is called an holomorphic section if it is tangent to a complex curve through that point. If the *sectional curvature* at a point is the same for all holomorphic sections at that point then the Riemann-Christoffel tensor can be written in the form

(6) $$R_{a\bar{b}c\bar{d}} = -\tfrac{1}{2}\mathrm{b}(g_{a\bar{b}}\,g_{c\bar{d}} \pm g_{a\bar{d}}\,g_{c\bar{b}}),$$

where b is the curvature on holomorphic sections; moreover b is a constant on the manifold if at each point the sectional curvature is the same for all holomorphic sections at that point [1, p. 184]. For a more thorough-going exposition of this subject matter, one is referred to the papers of Bochner [1; 2].

Bochner has shown that relation (6) implies the existence of an analytic coordinate system in which the line element has the form

(7) $$ds^2 = \frac{2\sum_a |dz_a|^2 + \mathrm{b}\{\sum_a |z_a|^2 \sum_b |dz_b|^2 - |\sum_a \bar{z}_a dz_a|^2\}}{(1 + \tfrac{1}{2}\mathrm{b}\sum_a |z_a|^2)^2}$$

The line element (7) is, for $\mathrm{b} > 0$, the Fubini-Study line element for complex projective space. If $\mathrm{b} = -2$ then (7) is the invarient line element of the unit cell $|z_1|^2 + \ldots + |z_n|^2 < 1$ under the group of all linear fractional transformations into itself. Let us call the complex projective space P^* and the unit cell E^* when they have these line elements.

We can now state our theorem.

THEOREM I. *If a complex manifold S has an Hermitian metric satisfying the Kaehler restriction and if it has constant holomorphic curvature and if the space is complete in this metric then its universal covering space is analytically isometrically equivalent with E^* for $\mathrm{b} = -2$ and with P^* for $\mathrm{b} > 0$.*

Let us assume from the start that S is simply connected. From Bochner's result we know that there is a local coordinate neighbourhood of each point of S such that the line element can be put into the form (7). Since S is complete there is a $\delta > 0$ such that each coordinate neighbourhood can be chosen with radius $> 2\delta$. Let $K(x, \delta)$ denote the cell of radius δ and centre x in S and let $L(y, \delta)$, denote the cell of radius δ and centre y in E^* or P^* according to whether $\mathrm{b} = -2$ or $\mathrm{b} > 0$. Then $K(x_0, \delta)$ can be mapped analytically isometrically onto $L(y_0, \delta)$ where x_0 is any point of S and y_0 is any point of E^*(or P^* as the case may be). Let x_1 be a point of $K(x_0, \delta)$ and let ϕ be the analytic isometry of $K(x_0, \delta)$ onto $L(y_0, \delta)$ considered, and let $y_1 = \phi(x_1)$. We now wish to show that ϕ can be extended to an analytic isometry of $K(x_0, \delta) \cup K(x_1, \delta)$ onto $L(y_0, \delta) \cup L(y_1, \delta)$.

So far we have discussed only the mapping of a $K(x, \delta)$ onto an $L(y, \delta)$, but we could also discuss the analytic isometry of a $K(x, 2\delta)$ onto an $L(y, 2\delta)$ since one of these is contained in a suitable coordinate neighbourhood (in which the line element has the form (7)) about each point. We now observe that there

exists a mapping of $K(x_0, 2\delta)$ onto $L(y_0, 2\delta)$ which agrees with ϕ on $K(x_0, \delta)$, for an isometry of $K(x_0, \delta)$ is characterized by the directions at y_0 into which the directions at x_0 are sent. Since ϕ is analytic it sends holomorphic directions into holomorphic directions. Now the unimodular unitary group acts transitively on the set of all holomorphic directions and we have the unimodular unitary group acting on both $K(x_0, 2\delta)$ and $L(y_0, 2\delta)$ (leaving invariant the metrics in each). Thus an analytic isometry ψ of $K(x_0, 2\delta)$ onto $L(y_0, 2\delta)$ can be made into one which agrees with ϕ on $K(x_0, \delta)$ by first performing ψ and then performing an unimodular transformation of $L(y_0, 2\delta)$ into itself. Therefore ϕ can be extended from $K(x_0, \delta)$ to $K(x_0, 2\delta)$ so in particular to $K(x_0, \delta) \cup K(x_1, \delta)$. Now let x'_0 be any point of S then we can continue (analytically and isometrically) the mapping ϕ to x'_0. To do this choose a path C with end points x_0 and x'_0; take points $x_1, \ldots, x_s$ on C so that $x_1 \in K(x_0, \delta)$, $x_2 \in K(x_1, \delta), \ldots, x'_0 \in K(x_s, \delta)$, then by the method described above ϕ can be extended to each cell in turn and so to x'_0. In this way we extend ϕ to all of S. But ϕ is single valued since S is simply connected. Now S satisfies all the conditions of being a covering space over part of E^* (or P^*), however since E^* (as well as P^*) is also simply connected ϕ is actually an homeomorphism of S into E^* (or P^*). Also ϕ is an analytic isometry by the way it was defined, and it must be onto since S is complete; thus we have the theorem.

Since we know that the complex projective space cannot cover, we have the following corollary:

COROLLARY. *If* $b > 0$ *then* S *must already be simply connected, hence* $S = P^*$.

On the other hand, E^* may well occur as the covering space of a non-simply connected S, even of a compact S. We now have

THEOREM II. *If* S *is a compact complex manifold with Hermite-Kaehler metric which has constant negative holomorphic curvature then the group of analytic homeomorphisms of* S *is finite.*

The proof of Theorem II follows quickly from Theorem I and [3, Theorem VII]. For S has E^* as universal covering space and E^* is a bounded domain in E^{2n} hence *a fortiori* a Picard domain. (In Theorem I we assumed that $b = -2$, but this was only so that E^* would be the *unit cell;* this is clearly inessential and $b < 0$ is all that matters.)

We now state: if S satisfies the conditions of Theorem II it uniformizes an algebraic variety. For let $\bar{S}$ be a fundamental domain for S in E^*, then $\bar{S}$ generates a discontinuous group Γ in E^* such that $E^* \pmod{\Gamma} = S$. Given such a group, where $\bar{S}$ is compact modulo Γ, Siegel has proved there exist n analytically independent automorphic functions in E^* relative to Γ [4, pp. 132–136]. To obtain an algebraic variety which S uniformizes we note that any $n + 1$ functions on E^*, automorphic relative to Γ, satisfy an algebraic relation [4, pp. 137–145]. Making this polynomial homogeneous we obtain a projective model of our algebraic variety.

Finally one notes that for S compact and $b > 0$ or $b < 0$ we have the existence of n analytically independent functions meromorphic on S. But for $b = 0$ we have no information as to the existence of functions. Indeed in this case S is a multi-torus which may have any number from $0, 1, \ldots, n$ analytically independent non-constant meromorphic functions. This phenomenon arises because of the need for *period relations* in the theory of Abelian functions; which in turn arises from the fact that in some multi-tori a local sub-variety is not part of a sub-variety in the large, not because the local sub-variety cannot be extended (it can be continued indefinitely), but because it winds infinitely often around the multi-torus and never meets itself (much as the familiar everywhere-dense integral curve winds on the two-dimensional torus). In fact a multi-torus (in n complex variables, $n > 1$) may be constructed which has no proper sub-varieties other than points. (For an example see [4, pp. 104–106].)

In concluding I wish to thank Professor S. Bochner for suggesting Theorem I. It is also explicitly stated in [2, p. 21].

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Louisiana State University

HARMONIC p-TENSORS ON NORMAL HYPERBOLIC RIEMANNIAN SPACES

G. F. D. DUFF

Introduction. The subject of this paper is the study of boundary value theorems for harmonic p-tensors on a Riemannian space with an indefinite metric of the normal hyperbolic signature. The p-tensors or p-vectors $\phi_{i_1 \ldots i_p}$ are alternating covariant tensors of rank p, which are closely related to differential forms ϕ of degree p, $p \leqslant m$, on an m-dimensional manifold.

On a Riemannian space with positive definite metric, the harmonic p-tensors have been studied by Hodge, de Rham, Kodaira, and others, and a theory which generalizes the classical potential theory, and possesses in addition certain new features, has been developed.

The present paper continues a program of extending the boundary value theorems of the theory of partial differential equations to p-tensors. In the positive definite (elliptic) case, boundary value theorems have been given [3] for the Beltrami-Laplace equation

$$\Delta\phi = 0$$

and for the harmonic field equations

$$d\phi = 0, \quad \delta\phi = 0.$$

These equations are now investigated under an indefinite metric.

The principal result is the solution of the Cauchy problem for the Beltrami-Laplace equation. After a preliminary section, the auxiliary conditions for the Cauchy problem are formulated, and uniqueness under these conditions is established. To construct the solution, I use the method of Riesz potentials. For the sake of brevity, reference to Riesz' paper [9] has been made wherever possible. The solution of the Beltrami-Laplace equation thus obtained is then applied to the construction of solutions of the harmonic field equations. In a concluding section, some special cases, in particular the electromagnetic field equations, are examined, and the main theorems are compared with the results which hold in the elliptic case.

1. Preliminaries. Let M be an m-dimensional orientable Riemannian space, with metric

$$ds^2 = g_{ij}\,dx^i\,dx^j$$

having the Lorentz signature. That is, on being transformed to a sum of squares,

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ds^2 contains one positive and m-1 negative terms. For convenience we shall refer to displacements as timelike if $ds^2 > 0$ and spacelike if $ds^2 < 0$. We shall also distinguish the timelike coordinate, say x^m, by the letter t in certain cases. This metric is also known as a normal hyperbolic metric (Hadamard), the corresponding Laplace equation being of the normal hyperbolic type. During most of this paper we assume that the metric is analytic, but this restriction can be removed.

Throughout we shall use such terms as continuous, compact, convergent, on the understanding that they refer to a suitable positive definite distance, such as the Euclidean distance.

We introduce on M the skew symmetric covariant tensors $\phi_{i_1 \dots i_p}$; corresponding to these are differential forms of degree p:

$$(1.1)\quad \phi = \phi_p = \sum_{i_1 < \dots < i_p} \phi_{i_1 \dots i_p}\, dx^{i_1} \wedge \dots \wedge dx^{i_p} = \phi_{(i_1 \dots i_p)}\, dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

The bracket enclosing the group of indices shall mean that the indices inside it are arranged in strictly increasing order. The differentials dx^i are multiplied together by the exterior multiplication indicated by the sign $\wedge$; hence these differentials anticommute. We now define the differential operator d;

$$(1.2)\qquad d\phi = (d\phi_{(i_1 \dots i_p)})\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

Thus $d\phi$ is a differential form of degree $p + 1$. Let

$$(1.3)\qquad \Gamma_{i_1 \dots i_p;\, j_1 \dots j_p} = \begin{vmatrix} g_{i_1 j_1} & \dots & g_{i_1 j_p} \\ \vdots & & \vdots \\ g_{i_p j_1} & \dots & g_{i_p j_p} \end{vmatrix},$$

then

$$\Gamma_{i_1 \dots i_p}{}^{j_1 \dots j_p}$$

is just the Kronecker symbol often denoted by

$$\delta_{i_1 \dots i_p}{}^{j_1 \dots j_p}.$$

Also let

$$(1.4)\qquad e_{i_1 i_2 \dots i_n} = \Gamma_{i_1 \dots i_n}{}^{12 \dots n}\, |\Gamma_{12 \dots n,\, 12 \dots n}|^{\frac{1}{2}}.$$

We denote covariant derivatives of a p-tensor

$$\phi_{i_1 \dots i_p} \text{ by } D_i\, \phi_{i_1 \dots i_p},$$

and set $D^i = g^{ij} D_j$. Thus

$$(1.5)\qquad D_i \phi_{i_1 \dots i_p} = \frac{\partial \phi_{i_1 \dots i_p}}{\partial x^i} - \sum_{k=1}^{p} \left\{ \begin{matrix} h \\ i_k\, i \end{matrix} \right\} \phi_{i_1 \dots i_{k-1}\, h\, i_{k+1} \dots i_p}.$$

In the notation of [8a],

$$(1.6)\qquad \begin{aligned} (d\phi)_{i_1 \dots i_{p+1}} &= \Gamma_{i_1 \dots i_{p+1}}{}^{j(j_1 \dots j_p)} D_j \phi_{(j_1 \dots j_p)} \\ &= \Gamma_{i_1 \dots i_{p+1}}{}^{j(j_1 \dots j_p)} \frac{\partial}{\partial x^j}\, \phi_{(j_1 \dots j_p)}. \end{aligned}$$

The operator $d.d$ is identically zero. If $d\phi = 0$, ϕ is said to be closed; if $\phi = d\chi$, ϕ is said to be derived, and is therefore closed. The converse statement that if ϕ is closed then ϕ is derived, is true "locally," but not in the large.

The dual $*\phi$ of a p-form ϕ is an $(m - p)$-form whose components are defined by the formula

$$(*\phi)_{j_1 \dots j_{m-p}} = e_{(i_1 \dots i_p) j_1 \dots j_{m-p}} \phi^{(i_1 \dots i_p)} \tag{1.7}$$

If p is the degree of ϕ, then

$$**\phi = (-1)^{mp+p}\phi.$$

Hence, save for a sign, the $*$ operation is its own inverse. It will be seen that ϕ and $*\phi$ are "perpendicular" forms at each point.

The operation of dual derivation is given by δ, where

$$\begin{aligned}(\delta\phi)_{i_1 \dots i_{p-1}} &= (-1)^{mp+m+1}(*d*\phi)_{i_1 \dots i_{p-1}} \\ &= -\Gamma_{ti_1 \dots i_{p-1}}{}^{(j_1 \dots j_p)} D^t \phi_{(j_1 \dots j_p)}\end{aligned} \tag{1.8}$$

It follows from the preceding remarks that $\delta.\delta = 0$. If $\delta\phi = 0$, ϕ is said to be coclosed, and if $\phi = \delta\chi$, ϕ is coderived. A p-tensor ϕ which is both closed and coclosed will be known as a harmonic field [6].

The Beltrami-Laplace (B.L.) operator Δ for p-tensors is given by

$$-\Delta = d\delta + \delta d, \tag{1.9}$$

and in expanded form may be written

$$(\Delta\phi)_{i_1 \dots i_p} = D^t D_t \phi_{i_1 \dots i_p} - \sum_{n=1}^{p} \Gamma_{k\ i_1 \dots i_p}{}^{t(j_1 \dots j_p)}\, g^{kj} R^h{}_{j_n j t} \phi_{j_1 \dots j_{n-1} h j_{n+1} \dots j_p} \tag{1.10}$$

where $R^h{}_{tjk}$ is the Riemannian curvature tensor and summation over the indices i, j, h, k and $(j_1 \dots j_p)$ is understood. If ϕ is a scalar, then $\Delta\phi$ reduces to the usual Laplacian. A p-tensor ϕ which satisfies the B.L. equation

$$\Delta\phi = 0 \tag{1.11}$$

is said to be a harmonic form.

From (10) it is clear that the B.L. equation is of the normal hyperbolic type under our Lorentzian metric. In fact, the equation (1.11) stands for a system of $\binom{m}{p}$ equations, one for each component of ϕ. Note that each of these equations has the same principal part, namely,

$$g^{ij} \frac{\partial^2 \phi_{i_1 i_2 \dots i_p}}{\partial x^i\, \partial x^j}.$$

Actually all terms containing first or second derivatives are of the same form in each component equation. Furthermore the form of the second order (principal) terms is independent of p, so the theory of the characteristics will be carried over unchanged from the scalar case. The characteristic surfaces of each component equation being the same, we may speak of a characteristic surface of (11). These surfaces are given by

$$C(x^1 \ldots x^m) = C(x) = 0,$$

where

$$g^{ij} \frac{\partial C}{\partial x^i} \frac{\partial C}{\partial x^j} = 0. \tag{1.12}$$

Let O be an arbitrary point of M which we select as origin of coordinates. The geodesics of zero length through O form a conoid [5; 8] which is a characteristic surface (1.12).

Let S be a sufficiently differentiable spacelike "initial" surface. For convenience we assume that S is compact. We construct a system of geodesic normal coordinates in which S is the hyperplane $x^m = t = 0$. Since the normals to S are timelike, we can write

$$ds^2 = dt^2 - g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{1.13}$$

where the Greek indices range from 1 to $m - 1$. In this system [10], $g_{tm} = \delta_m{}^t$, $D_m \equiv D^m$.

Associated with S we define a region R as follows. R shall consist of those points P on the positive side ($t \geqslant 0$) of S, such that the retrograde characteristic cones (null cones) C_P with vertex P, together with that part of S intercepted by C_P, bound a simply-connected region of M. Denoting this region by $D_S{}^P$, following Riesz, we see that if $P \in R$, every point of $D_S{}^P$ also belongs to R. We may refer to $R = R_S$ as the region of exclusive dependence upon S, in the sense that, as will be shown, solutions of the B.L. equation with data assigned on S are determined throughout R.

Thus defined, R_S may contain non-bounding p-cycles. Any p-cycle of R is however homologous to a p-cycle of S, since a continuous cylindrical $(p + 1)$-dimensional surface can be constructed on the cycle, lying in R and joining the cycle to a (homologous) p-cycle which is its intersection with S. An analogous remark holds for the relative p-cycles of R (mod S): these can all be deformed into S and are therefore zero.

A p-form ϕ induces on any surface S (supposed given by $x^m = 0$) a p-form $t\phi$, the components of which are precisely those components of ϕ which are not multiplied by dx^m. The form $t\phi$ is known as the tangential boundary component of ϕ on S. The residual part of ϕ, which contains the factor dx^m, is known as the normal boundary component and is denoted by $n\phi$. Thus

$$\phi = t\phi + n\phi = t\phi + \phi_1 \wedge dx^m. \tag{1.14}$$

If $p = 0$, $t\phi \equiv \phi$; if $p = m$, $t\phi \equiv 0$. On S, and only there, this division into tangential and normal components has an invariant meaning. From the properties of the dual operator (7) it follows that the commutation rules

$$*t = n*, \quad *n = t* \tag{1.15}$$

hold. Also we note that if ϕ is assigned on S (both $t\phi$ and $n\phi$), then $td\phi$ and $n\delta\phi$ are thereby determined. However, $nd\phi$ and $t\delta\phi$ can be assigned independently,

since each component of these forms contains a normal derivative of a component of ϕ. Note especially that $nd\phi$ and $t\delta\phi$ have $\binom{m-1}{p}$ and $\binom{m-1}{p-1}$ components respectively, making a total of $\binom{m}{p}$ components. Finally, we easily verify that

$$td\phi = d_S t\phi, \tag{1.16}$$

where d_S is the differential operator (6) in the surface S.

The surface S will be the carrier of the "initial" data for the Cauchy problem for the p-tensor B.L. equation. Since this equation is of the second order, it is to be expected that values of the components of ϕ and of their normal derivatives will be assigned on S. In order to express the data on the surface S in an invariant form, we shall use the following lemma.

LEMMA 1. *Let S be a surface whose equation is $x^m = 0$ in a sufficiently differentiable system of coordinates in M, and let ϕ be a p-form defined in a neighbourhood of S. Then the specification of any one of the following sets of data is equivalent:*

$$\text{(a) } \phi, nd\phi, t\delta\phi; \quad \text{(b) } \phi, D_m\phi, \quad \text{(c) } \phi, \frac{\partial\phi}{\partial x^m}. \tag{1.17}$$

The vanishing of one implies the vanishing of the other two.

Proof of the lemma will be given in cyclic order. First we show that knowledge of (a) enables us to calculate (b). If $i_1 \dots i_p$ are all less than m, then from (6) follows the value of

$$D_m\phi_{i_1 \dots i_p}$$

in terms of

$$(d\phi)_{i_1 \dots i_p m}$$

and covariant derivatives of ϕ along the surface. Similarly, if $i_p = m$, say

$$D_m\phi_{i_1 \dots i_{p-1} m}$$

is given by (8) in terms of data included in (a). Next, (b) clearly implies (c), since all components of ϕ are known on S. Finally, (c) implies that $d\phi$ is known on S, by (6). Since $*\phi$ and $D_m*\phi$ are obtainable from (c), $\delta\phi$ is also known on S. This proves the equivalence. The last statement of the lemma is evident.

Let C be a $(p+1)$-chain of M with real coefficients, and let its boundary be denoted by bC. Then Stokes's formula

$$\int_C d\phi = \int_{bC} \phi \tag{1.18}$$

holds. Hence, if ϕ is closed, its integral over a bounding cycle C vanishes. It follows that a closed p-form ϕ has periods $\int_Z \phi$ on p-cycles Z, which depend only on the homology class of Z. In a manifold with boundary [**2**; **8b**], a closed p-form ϕ is derived if and only if its periods (on absolute cycles) vanish.

If ϕ and ψ are two forms, the sum of whose degrees does not exceed m, we have

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi, \tag{1.19}$$

where p is the degree of ϕ.

Consider a subregion of M, which we may take to be the region R. Let α and β be two forms of degree p defined in R, and let

$$(\alpha, \beta)_R = \int_R \alpha \wedge *\beta = (\beta, \alpha)_R \tag{1.20}$$

denote the scalar product over R. We remark that in the elliptic case ($ds^2 > 0$), the scalar square $(\alpha, \alpha)_R$ is positive definite; however this no longer holds for our metric. However, for any two forms ϕ and ψ of degrees p and $(p+1)$ respectively, we have the formula of Green:

$$(d\phi, \psi)_R - (\phi, \delta\psi)_R = \int_{bR} \phi \wedge *\psi, \tag{1.21}$$

which follows from (18), (19), and (20). The metric being indefinite, we should change the sign of the surface integrals over any timelike surface. However, for all applications which we have in view, the surface bR will consist of spacelike surfaces and null cones, and the integrand will be made to vanish on the latter; so the formula will apply as written. From (21) we obtain by formal transformation the extended formula of Green, namely,

$$-(\Delta\phi, \psi)_R + (\phi, \Delta\psi)_R = \int_{bR} (\phi \wedge *d\psi - \psi \wedge *d\phi + \delta\phi \wedge *\psi - \delta\psi \wedge *\phi), \tag{1.22}$$

where ϕ and ψ are now of equal degree. Regarding the right-hand side of (22) as a linear functional of the form ϕ, ψ being held fixed, we observe that the successive terms contain $t\phi$, $nd\phi$, $t\delta\phi$, and $n\phi$; and that each of these expressions can be assigned independently of the other three. All four together are equivalent to any one of (17).

2. A uniqueness property of the B.L. equation. Let $R = R_S$ be the region of dependence associated with an initial surface S as described in the preceding section. In terms of the geodesic normal coordinate system (13), we may write the partial differential equation (11) in the form

$$L(\phi_{i_1 \ldots i_p}) \equiv \frac{\partial^2 \phi_{i_1 \ldots i_p}}{\partial t^2} - g^{\alpha\beta} \frac{\partial^2 \phi_{i_1 \ldots i_p}}{\partial x^\alpha \partial x^\beta} + a_{i_1 \ldots i_p}{}^{k(j_1 \ldots j_p)} \frac{\partial}{\partial x^k} \phi_{(j_1 \ldots j_p)} + b_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)} \phi_{(j_1 \ldots j_p)} = 0, \tag{2.1}$$

where $t = x^m$, $i \leqslant \alpha, \beta \leqslant m-1$; and

$$a_{i_1 \ldots i_p}{}^{k(j_1 \ldots j_p)}, \quad b_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)}$$

are polynomial functions of the components of the metric tensor and their first derivatives, which are skew-symmetric with respect to the indices $i_1 \ldots i_p$;

$j_1 \ldots j_p$. In this coordinate system the Cauchy problem consists of solving (2.1) (in general with a non-homogeneous right-hand term) given the values of the

$$\phi_{i_1 \ldots i_p} \text{ and } \frac{\partial}{\partial t}\phi_{i_1 \ldots i_p}$$

on S.

LEMMA II. *The solution of the Cauchy problem for* (2.1) *is unique in* R_s.

The argument which we use is of a standard type [1, II, p. 310]. Let P be a point of R with retrograde null cone C_P, and suppose ϕ and $\partial\phi/\partial t$ are assigned the value zero on the section S_P of S enclosed by C_P. We have to prove that $\phi(P)$ is zero. Let ϕ_i denote a typical component of ϕ. For $Q \in R$, we have

$$\phi_i(Q) = \int_0^{t(Q)} \frac{\partial \phi_i}{\partial t}\, dt,$$

so that by the Schwarz inequality

$$\phi_i^2(Q) \leqslant t(Q) \int_0^{t(Q)} \left(\frac{\partial \phi_i}{\partial t}\right)^2 dt = t \int_0^t \phi_{ii}^2 dt,$$

where $t = t(Q)$ and the integration is taken along a parametric line of the coordinate system.

The subscripts α and t will denote partial derivatives with respect to x^α and t, respectively. Let D_h be the region enclosed by C_P, and the planes $t = 0$, $t = h$, and $S_P(h)$ that part of the plane $t = h$ cut off by C_P. We have

$$\begin{aligned} \int_{S_P(h)} \phi_i^2(Q)\, dA &\leqslant h \int_{S_P(h)} dA \int_0^h \phi_{ii}^2\, dt \\ &\leqslant h \int_{D_h} \phi_{ii}^2\, dV \leqslant h \int_0^h E_i(h)\, dh \end{aligned} \tag{2.2}$$

where

$$E_i(h) = \int_{S_h} (\phi_{ii}^2 + g^{\alpha\beta}\phi_{i\alpha}\phi_{i\beta})\, dA. \tag{2.3}$$

This follows since the metric form $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ in the spacelike surface S_h is positive definite. Integrating (2.2) with respect to t from 0 to h, and noting that $E_i(h) \geqslant 0$, we have

$$\int_{D_h} \phi_i^2\, dV \leqslant h^2 \int_0^h E_i(t)\, dt. \tag{2.4}$$

We next consider the identity

$$\begin{aligned} 2\phi_{ii}L(\phi_i) &= 2\phi_{ii}\phi_{iii} - 2g^{\alpha\beta}\phi_{i\alpha\beta}\phi_{ii} + B_i(\phi_j, \phi_{j\alpha}, \phi_{jt}) \\ &= (\phi_{ii})_t^2 + (g^{\alpha\beta}\phi_{i\alpha}\phi_{i\beta})_t - 2(g^{\alpha\beta}\phi_{i\alpha}\phi_{ii})_\beta + B_i(\phi_j, \phi_{j\alpha}, \phi_{jt}), \end{aligned} \tag{2.5}$$

where B_i denotes an expression linear in the components of ϕ and their first derivatives, multiplied by ϕ_{ii}. Let α_ν and t_ν be the direction numbers ds/dx^α, ds/dt of the conormal ν to a surface. On the surface C_P we have $ds^2 = 0$, whence

$$t_\nu^2 = g^{\alpha\beta}\alpha_\nu\beta_\nu \qquad \text{on } C_P. \tag{2.6}$$

We integrate (2.5) over D_h and apply the divergence theorem. Since $L(\phi_i)$ is zero we have

$$0 = \int_{bD_h}\left\{(\phi_{i1})^2 t_\nu + (g^{\alpha\beta}\phi_{i\alpha}\phi_{i\beta})t_\nu - 2(g^{\alpha\beta}\phi_{i\alpha}\phi_{i1})\beta_\nu\right\} dA + \int_{D_h} B_i\, dV.$$

On S_P ($t = 0$), ϕ and its derivatives are to be zero; on S_h we have $t_\nu = 1$, $\alpha_\nu = 0$. Using (2.6) we then find

$$\begin{aligned}\int_{C_P(h)} \frac{1}{t_\nu}\left\{g^{\alpha\beta}\alpha_\nu\beta_\nu\phi_{i1}^2 - 2t_\nu g^{\alpha\beta}\beta_\nu\phi_{i\alpha}\phi_{i1} + g^{\alpha\beta}t_\nu^2\phi_{i\alpha}\phi_{i\beta}\right\} dA \\ + \int_{S_h}\left\{\phi_{i1}^2 + g^{\alpha\beta}\phi_{i\alpha}\phi_{i\beta}\right\} dA \\ = \int_{C_P(h)} \frac{1}{t_\nu}\sum_{\alpha\beta} g^{\alpha\beta}(t_\nu\phi_{i\alpha} - \alpha_\nu\phi_{i1})(t_\nu\phi_{i\beta} - \beta_\nu\phi_{i1})\, dA + E_i(h) \\ = -\int_{D_h} B_i\, dV = R_i,\end{aligned}$$

denoting the value of the last integral by R_i. Here we have denoted by $C_P(h)$ that portion of C_P which lies between the planes $t = 0$, $t = h$. On C_P, t_ν is positive. Recalling that the metric form $g_{\alpha\beta}\,dx^\alpha\,dx^\beta$ is positive definite, we conclude that

$$E_i(h) \leqslant R_i. \tag{2.7}$$

Adding together these relations for all components ϕ_i, we find

$$\Sigma(h) = \sum_i E_i(h) \leqslant \sum_i R_i = R. \tag{2.8}$$

Now the quantity R defined by (2.8) is an integral over D_h of terms $\phi_{i\alpha}\phi_{j\beta}$ or $\phi_j\phi_{i\alpha}$, each of which is less than half the sum of the squares of the two factors. The coefficients of these terms in the integral R are bounded. Hence, in view of (2.4) we have an inequality

$$R \leqslant K\int_0^h \Sigma(h)\, dh, \tag{2.9}$$

for some positive constant K. From (2.8) and (2.9) follows

$$\Sigma(h) \leqslant K\int_0^h \Sigma(t)\, dt \leqslant K\int_0^k \Sigma(t)\, dt$$

for every $k \geqslant h$. Integrating between 0 and h,

$$\int_0^h \Sigma(h)\, dh \leqslant Kh\int_0^h \Sigma(h)\, dh. \tag{2.10}$$

But (2.10) is manifestly false for $h < 1/K$, unless $\Sigma(h) \equiv 0$, $(0 \leqslant h \leqslant 1/K)$. Hence $\Sigma(h)$ must vanish in this range, so that all derivatives of components of ϕ vanish for $t < 1/K$. Hence ϕ itself is zero in this interval of values of t. Since

K can be chosen to depend only on S and P and the metric, we can repeat the above process starting from the surface $S_{1/K}$, and so on. Hence finally we obtain $\phi(P) = 0$. This completes the uniqueness proof.

3. The Riesz kernel. A detailed exposition of the solution of the Cauchy problem for normal hyperbolic equations in Riemannian spaces has been given by M. Riesz [9]. We shall construct the solution for the p-tensor B.L. equation using Riesz's method. For a complete discussion of matters upon which we only touch, the reader is referred to Riesz's work.

It is required to construct a double p-tensor kernel

$$V^\alpha(x, y) = V^\alpha_p(x, y) = \left\{ V^\alpha_p(x, y)_{i_1 \ldots i_p;\ j_1 \ldots j_p} \right\} \tag{3.1}$$

having the properties:

(a) $\Delta_x V^{\alpha+2} = V^\alpha$,

(b) $V^\alpha(x, y) = 0(s^{\alpha-m})$,

where s is the geodesic distance from x to y,

(c) $V^\alpha(x, y) = V^\alpha(y, x)$,

(d) $\int_{D_y^z} V^\alpha(x, y) \wedge {}^*V^\beta(z, y) = V^{\alpha+\beta}(x, y)$,

where D_y^z is the double conoid enclosed by the retrograde null cone of one argument point and the direct cone of the other,

(e) $V^\alpha(x, y)|_{y=z} = \sum_{i_1 < \ldots < i_p} \frac{1}{2^{m-1}\pi^{\frac{1}{2}(m-1)}}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} dy^{i_1} \wedge \ldots \wedge dy^{i_p}$.

We proceed to construct V^α by means of a series expansion in powers of s. Properties (c) and (d) will be established later.

Certain facts from the theory of geodesics, which we now set forth, will be required. Riesz gives a detailed account of these matters. Choosing the point x_0 as origin, we denote by $x^1 \ldots x^m$ a coordinate system in M. On a family of curves C, let $p_1 \ldots p_m$ be the conjugate variables defined by

$$p_i = g_{ik}\dot{x}^k, \quad \dot{x}^k = \frac{dx^k}{d\sigma}, \tag{3.2}$$

where σ is a parameter along C. We introduce the Hamiltonian function $H(x_0, x)$, where

$$2H(x_0, x) = g^{ik} p_i\, p_k, \tag{3.3}$$

and note that the differential equations of the geodesics can be written

$$\dot{x}^i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{3.4}$$

The geodesics passing through x_0 are determined by the quantities $\dot{x}^i_0$. As we

shall need to consider only timelike geodesics ($ds^2 > 0$), we may take s as proportional to the parameter σ on the curve. If we set $\sigma(x) = 1$, then since H is constant along a geodesic we have

$$s = s(x_0, x) = \int_0^1 \sqrt{g_{ik}\dot{x}^i\dot{x}^k}\, d\sigma = \int_0^1 \sqrt{g^{ik}p_ip_k}\, d\sigma = \int_0^1 \sqrt{2H}\, d\sigma = \sqrt{2H}; \tag{3.5}$$

letting $P = s^2$ we find $P = 2H$.

We now select normal coordinates ξ^k which determine [9, p. 173] the point x:

$$\xi^k = \sigma_x\, \dot{x}_0^k = s\, \xi_0^k. \tag{3.6}$$

Taking S as parameter, we have $\xi^k = \xi_0^k$ along the geodesic arc.

A number of useful identities may now be derived. Since H is constant along a geodesic, we have

$$g^{ik}(0)\eta_i\,\eta_k = g^{ik}(x)\eta_i\,\eta_k \tag{3.7}$$

where the η_i are variables conjugate to the ξ^k. From (3.7) follows in view of (3.2) the relation.

$$\begin{aligned} g_{ik}(0)\xi^i\xi^k &= g_{ik}(0)s^2(0,\xi)\xi_0^i\xi_0^k \\ &= g_{ik}(\xi)s^2(0,\xi)\xi^i\xi^k = g_{ik}(\xi)\xi^i\xi^k; \end{aligned} \tag{3.8}$$

hence from (3.5)

$$P(0,\xi) = s^2(0,\xi) = g_{ik}(0)\xi^i\xi^k = g_{ik}(\xi)\xi^i\xi^k. \tag{3.9}$$

Thus, choosing Riemannian coordinates at x_0 such that $g_{mm} = 1$, $g_{kk} = -1$, $k \neq m$, $g^{ik} = 0$, $i \neq k$, we find for P the Lorentz distance

$$P = s^2 = (\xi^m)^2 - (\xi^{m-1})^2 - \ldots - (\xi^1)^2. \tag{3.10}$$

Next, denoting $\partial P/\partial\xi^i$ by P_i, we have

$$P_i = 2g_{ij}\xi^j,\quad P^i = 2\xi^i,$$

whence

$$P^iP_i = 4g_{ij}\xi^i\xi^j = 4P. \tag{3.11}$$

This partial differential equation for P shows, that, as has been mentioned, the geodesics of zero length through x_0 form a characteristic surface for the B.L. equation. In fact, the geodesic lines are just the bicharacteristics.

Again, let V be a differentiable function of position; then

$$P^i\frac{\partial U}{\partial\xi^i} = 2\xi^i\frac{\partial U}{\partial\xi^i} = 2s\xi^i\frac{\partial U}{\partial\xi^i} = 2s\frac{d\xi^i}{ds}\frac{\partial U}{\partial\xi^i} = 2s\frac{dU}{ds}. \tag{3.12}$$

Finally, we note that

$$\begin{aligned} g^{ij}\frac{\partial^2 P}{\partial\xi^i\partial\xi^j} &= 2g^{ij}g_{ij} + 2g^{ij}\xi^k\frac{\partial g_{jk}}{\partial\xi^i} \\ &= 2m + 0(s^2). \end{aligned} \tag{3.13}$$

These formulae will now be applied to the construction of the kernel. We shall need to calculate the Laplacian of a p-tensor ϕ multiplied by a power of the geodesic distance. Let $F(P)$ be a scalar function of P. Then we have

$$\begin{aligned}\Delta(F(P)\phi) &= D^i D_i(F(P)\phi) + BF(P)\phi \\ &= F(P)\Delta\phi + (D^i D_i F(P) + 2D^i F(P) D_i)\phi,\end{aligned} \tag{3.14}$$

where B denotes a suitable multiplying matrix. We find

$$\begin{aligned}D^i D_i F(P) &= D^i(F'(P)P_i) \\ &= g^{ik}[F''(P)P_iP_k + F'(P)P_{ik} - \{{}^r_{ik}\}F'(P)P_r] \\ &= 4PF''(P) + 2mF'(P) + F'(P)sA_1,\end{aligned} \tag{3.15}$$

where A_1 is a suitable bounded multiplying matrix. For the last term in (3.14) we find

$$\begin{aligned}&D^i F(P) D_i \phi_{(j_1 \dots j_p)} \\ &= F'(P)\left[P^i \frac{\partial}{\partial \xi^i}\phi_{(j_1 \dots j_p)} - P^i \sum_{\alpha=1}^{p} \{{}^h_{i_\alpha i}\}\phi_{j_1 \dots j_{\alpha-1} h j_{\alpha+1} \dots j_p}\right] \\ &= F'(P)\left[2s\frac{d\phi_{(j_1 \dots j_p)}}{ds} + sA_{2(j_1 \dots j_p)}\phi\right]\end{aligned} \tag{3.16}$$

where A_2 is a second matrix with bounded elements near x_0.

In the preceding equations, ϕ is to be understood as a vector with $\binom{m}{p}$ components, while A_1, A_2 and B are square matrices of order $\binom{m}{p}$. Continuing with this notation, we have

$$\begin{aligned}\Delta(F(P)\phi) &= F(P)\Delta\phi + [4PF''(P) + 2mF'(P)]\phi \\ &\quad + sF'(P)A\phi + 4F'(P)s\frac{d\phi}{ds},\end{aligned} \tag{3.17}$$

where $A = A_1 + A_2$ is also bounded, the bound depending solely upon the bounds of components of the metric tensor and their first derivatives.

Following Riesz, we set

$$V^\alpha(0, \xi) = \sum_{k=0}^{\infty} \frac{s^{\alpha-m+2k}V_k}{H_m(\alpha, k)} = \frac{s^{\alpha-m}}{K_m(\alpha)}\sum_{k=0}^{\infty}\frac{s^{2k}V_k(0, \xi)}{L_m(\alpha + 2k)}, \tag{3.18}$$

and determine the double (matrix) forms V_k successively so that (a) shall hold. Since $P = s^2$, we have

$$\begin{aligned}\Delta V^{\alpha+2} &= \sum_{k=0}^{\infty}\frac{\Delta s^{\alpha+m+2k+2}V_k}{K_m(\alpha+2)L_m(\alpha+2k)} \\ &= \sum_{k=0}^{\infty}\frac{s^{\alpha-m+2k}}{H_m(\alpha+2k)}\Big(s^2\Delta V_k + (\alpha - m + 2k + 2)(\alpha + 2k)V_k \\ &\quad + 2(\alpha - m + 2k + 2)sAV_k + 2(\alpha - m + 2k + 2)s\frac{dV_k}{ds}\Big)\end{aligned} \tag{3.19}$$

$$= \frac{s^{\alpha-m}}{K_m(\alpha+2)} \sum_{k=0}^{\infty} s^{2k} \left(\frac{\Delta V_{k-1}}{L_m(\alpha+2k)} + \frac{\alpha-m+2k+2}{L_m(\alpha+2k+2)} \left[2s \frac{dV_k}{ds} + (\alpha+2k) V_k + 2sA V_k \right] \right).$$

We now choose

$$L_m(\alpha+2) = (\alpha+2-m)\, L_m(\alpha), \tag{3.20}$$

and determine the V_k by the recurrent system of differential equations

$$2s \frac{dV_k}{ds} + (2k+2sA) V_k + \Delta V_{k-1} = 0, \tag{3.21}$$

with

$$V_{-1} \equiv 0, \quad V_0(0,0) = I, \tag{3.22}$$

where I is the unit matrix of order $\binom{m}{p}$. The expression (3.19) now becomes

$$\Delta_k V^{\alpha+2} = \frac{\alpha}{K_m(\alpha+2)} \sum_{k=0}^{\infty} \frac{s^{\alpha-m+2k} V_k}{L_m(\alpha+2k)}$$

which is equal to V^α if we choose

$$K_m(\alpha+2) = \alpha K_m(\alpha). \tag{3.23}$$

From (3.20) and (3.23) we have

$$K_m(\alpha) = K 2^{\frac{1}{2}\alpha} \Gamma\left(\frac{\alpha}{2}\right); \quad L_m(\alpha) = L 2^{\frac{1}{2}\alpha} \Gamma\left(\frac{\alpha+2-m}{2}\right),$$

and we choose K, L so that (e) is satisfied:

$$H_m(\alpha, k) = \pi^{\frac{1}{2}(m-2)} 2^{\alpha+k-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+2k+2-m}{2}\right). \tag{3.24}$$

In these details we follow the scalar treatment.

We have now to prove the convergence of the series for V^α. The relation (3.21) can be written

$$\frac{d}{ds}(s^k V_k) + A\, s^k V_k = -\tfrac{1}{2} \Delta V_{k-1}\, s^{k-1}. \tag{3.25}$$

To solve (3.25) consider the adjoint system

$$\frac{d}{ds} Y = A' Y, \tag{3.26}$$

and let Y be a (nonsingular) matrix solution of (3.26) with $|Y(0)| = 1$. It follows from the Lyapunov relation

$$Y(s) = Y(0) \exp\left[\int_0^s \operatorname{tr} A\, ds \right]$$

that $Y(s)$ is nonsingular, hence $Y^{-1}(s)$ exists and is bounded for $|s| < 1$. Since $s^k V_k$ vanishes for $s = 0$, we find as solution of (3.25)

$$s^k V_k = -\tfrac{1}{2} Y^{-1}(s) \int_0^s Y(\sigma) \Delta V_{k-1}(\sigma) \sigma^{k-1} d\sigma. \tag{3.27}$$

With the vector and matrix norm

$$||X|| = \sqrt{\Sigma |X_i|^2}$$

(sum over all components), we have

$$s^k ||V_k|| \leqslant \tfrac{1}{2} ||Y^{-1}(s)|| \int_0^s ||Y(\sigma)||\, ||\Delta V_{k-1}(\sigma)|| \sigma^{k-1} d\sigma. \tag{3.28}$$

Let us assume that for suitable constants τ and k and for $|s| < \epsilon$,

$$||V_{k-1}|| < K^k \left(1 - \frac{\rho}{\tau}\right)^{-2k+1}; \rho = \sum_j |\xi^j|; \tag{3.29}$$

Then [5; 6],

$$||\Delta V_{k-1}|| < K^k K_1 \left(1 - \frac{\rho}{\tau}\right)^{-2k-2},$$

and from (3.28) we have

$$s^k ||V_k|| \leqslant \tfrac{1}{2} MK^k K_1 \int_0^s \frac{\sigma^{k-1} d\sigma}{(1 - \sigma\lambda/\tau)^{2k+2}} < s^k \frac{MK^k K_1}{2k} \left(1 - \frac{\rho}{\tau}\right)^{-2k-1}, \tag{3.30}$$

where λ and M are independent of k. Since an estimate (3.29) holds for V_0 it holds for V_1 and hence for all V_k. It follows from (3.30) that the series $\Sigma s^{2k} V_k$ converges geometrically for s sufficiently small. From (3.24) it follows that the series for V^α converges for all values of α if $s \neq 0$, is sufficiently small; and uniformly in any right half plane of α. Hence $V^\alpha(0, \xi)$ is defined as an analytic function of α, and regular for $R(\alpha) > m$ and $\xi \in R$ [cf 9; 5].

Exactly as in [9, p. 186] it follows that an estimate

$$||V^\alpha(x, y)|| \leqslant C s^{R(\alpha)-m} \left|\left(\frac{\alpha}{e}\right)^{-\alpha}\right| |\alpha|^q, \tag{3.31}$$

holds, where $R(\alpha) > m$, and C, q are independent of α.

We remark that the present construction follows that of Riesz very closely; the only additional features being that it was necessary to show that the matrix A was bounded and to solve a system of recursive equations for the V_k. For the non-analytic case Riesz has given a method of construction for the kernel, and the same formal considerations apply to the present p-tensor kernel. Since the details are lengthy we omit them.

4. Construction of the solution. We continue the adaptation of Riesz' method for the p-tensor B.L. equation. We now define the Riesz fractional potential which plays the central role in the solution of the Cauchy problem. Let

$$I^\alpha \phi(x) = (\phi(y), V^\alpha(x, y))_{D_x^x} = \int_{D_x^x} \phi(y) \wedge *V^\alpha(x, y), \; R(\alpha) > m - 2. \tag{4.1}$$

Here D_x^x is the conical region bounded by C_x and S, and ϕ is a p-form defined in R and assumed sufficiently differentiable for the prupose in hand. Riesz has

proved certain relations in the scalar case, which as we now show, can be extended to the present p-tensor treatment. If ϕ is of the class $C[\frac{1}{2}(m-1)]$, then

$$I^0\phi = \phi, \tag{4.2}$$

and in general,

$$\Delta I^\alpha \phi = I^{\alpha-2}\phi, \tag{4.3}$$

and

$$I^\alpha I^\beta \phi = I^{\alpha+\beta}\phi. \tag{4.4}$$

Riesz has also given a method of analytic continuation of $I^\alpha\phi$ into the left half-plane $R(\alpha) < m-2$.

First we shall prove (4.2). A typical component of $I^\alpha\phi$ is

$$\begin{aligned}(I^\alpha\phi)_{i_1\dots i_p} &= \int_{D_s^x} \sum_{k_1<\dots<k_p} \phi_{k_1\dots k_p}(y)\, V^\alpha_{i_1\dots i_p;\, k_1\dots k_p}\, dV_m \\ &= \sum_{k_1<\dots<k_p} \int_{D_s^x} \phi_{k_1\dots k_p}(y) \frac{s^{\alpha-m}}{H_m(\alpha,0)}\, v^\alpha_{i_1\dots i_p;\, k_1\dots k_p}(x,y)\, dV_m,\end{aligned} \tag{4.5}$$

where $v^\alpha(x, y)$ is a double p-form regular on the cone D_s^x and such that

$$v^\alpha(x,x) = V_0(x,x) = I_p. \tag{4.6}$$

The recurrence relation (3.25) for V_0 can be written

$$\frac{dV_0}{ds} + A\, V_0 = 0,$$

whence it follows that the off-diagonal elements of V_0, hence also those of $v^\alpha(x, y)$, are $O(s)$. The typical component (4.5) then breaks up into a sum of integrals of the type studied by Riesz. We may assume normal coordinates lending to the Lorentz distance (3.10). Using this coordinate system, it has been shown [4, 9] that an expression of the form

$$\lim_{\alpha\to 0} \int_{D_s^x} \frac{s^{\alpha-m}}{H_m(\alpha,0)}\, \phi(y)\, dV_m$$

depends only on the value of ϕ at the point x, and is in fact equal to $\phi(x)$. This result implies that as $\alpha \to 0$, all terms of (4.5) save the diagonal term tend to zero due to the factor $O(s)$, and that the remaining term is precisely $\phi_{i_1\dots i_p}(x)$. This establishes (4.2).

We now introduce the formula of Green (1.22) for the region D_s^x, and set $\psi = V^{\alpha+2}(x, y)$. We obtain

$$\begin{aligned}&(\Delta\phi,\, V^{\alpha+2})_{D_s^x} - (\phi,\, \Delta V^{\alpha+2})_{D_s^x} \\ &\quad = \int_{bD_s^x} (V^{\alpha+2} \wedge *d\phi - \phi \wedge *dV^{\alpha+2} + \delta V^{\alpha+2} \wedge *\phi - \delta\phi \wedge *V^{\alpha+2}).\end{aligned}$$

Supposing that $R(\alpha) > m$, the surface integrals over the cone vanish because of the factor $s^{\alpha+2-m}$ contained in $V^{\alpha+2}$, and we are left with the surface integrals over S^x. Supposing further that $\Delta\phi = \rho$ and replacing $\Delta V^{\alpha+2}$ by V^α, we find

$$(4.7)\quad I^{\alpha}\phi = I^{\alpha+2}\rho - \int_{s^*}(V^{\alpha+2}\wedge *d\phi - \phi\wedge *dV^{\alpha+2} + \delta V^{\alpha+2}\wedge *\phi - \delta\phi\wedge *V^{\alpha+2}).$$

To facilitate the study of (4.7), we write

$$(4.8)\quad \begin{aligned} J_1(\phi,\alpha) &= -\int_{s^*} V^{\alpha}\wedge *d\phi \\ J_2(\phi,\alpha) &= +\int_{s^*}\phi\wedge *dV^{\alpha}, \\ J_3(\phi,\alpha) &= -\int_{s^*}\delta V^{\alpha}\wedge *\phi, \\ J_4(\phi,\alpha) &= +\int_{s^*}\delta\phi\wedge *V^{\alpha}; \end{aligned}$$

then (4.7) may be written

$$(4.9)\quad I^{\alpha}\phi = I^{\alpha+2}\rho + \sum_{\mu=1}^{4} J_{\mu}(\phi,\alpha+2).$$

We now formulate our theorem.

THEOREM I. *Let S be a spacelike surface of the class $C[\frac{1}{2}(m+3)]$, and let R be its region of exclusive dependence. Let ξ be a p-form of the class $C[\frac{1}{2}(m+3)]$ defined in a neighbourhood of S (or let $t\xi$, $n\xi$, $nd\xi$, $t\delta\xi$ be given on S); and let ρ be a p-form of class $C[\frac{1}{2}(m+4)]$ in R. Then there exists in R a unique p-form ϕ such that*

$$(4.10)\quad \Delta\phi = \rho$$

in R, and

$$(4.11)\quad t\phi = t\xi, \quad n\phi = n\xi, \quad nd\phi = nd\xi, \quad t\delta\phi = t\delta\xi$$

on S. This form is given sufficiently close to S by

$$(4.12)\quad \phi = I^{0}\rho + \sum_{\mu=1}^{4} J_{\mu}(\xi, 2).$$

The main steps of the proof are as follows. It must first be shown that (4.9) can be continued analytically to $\alpha = 0$, so that (4.12) has a meaning. Then the conditions (4.10) and (4.11) are to be verified. The uniqueness has been established in Lemma II, so the proof will then be complete.

The analytical continuation has been carried through by Riesz [9] and by Fremberg [4, pp. 19-53]; since the essentials of the matter are in no way altered in the present case we shall not repeat the details. We remark that the continuation is possible if the forms ρ, ξ and the surface S have the differentiability properties stated in the theorem. It also follows that (4.12) is then continuous together with its first and second derivatives.

At $\alpha = 0$, the function $H_m(\alpha, 0)$ has a simple pole. Fremberg has also proved that the surface potentials can be continued to $\alpha = 0$, leading to finite expressions apart from the factor $1/H_m(\alpha, 0)$. It follows that the expressions J_{μ} vanish for $\alpha = 0$:

$$J_\mu(\xi, 0) = 0. \tag{4.13}$$

We must now establish a number of relations involving the B.L. operator and the integral I^α. First we make the important observation that in calculating derivatives with respect to x of I^α and the J_μ, we may disregard the variation at the limits of integration [9, pp. 67-70]. The justification of this is that for $R(\alpha) > m + k$, the kernel V^α is zero, together with its derivatives up to order k, on the cone D_x. It follows by analytic continuation that for all α we may calculate derivatives by formal differentiation under the integral sign. In view of condition (a) of §3, it therefore follows that (4.3) *holds whenever it has meaning.* In particular, if the continuation is valid to $\alpha = -2$, we have

$$\Delta\phi = I^{-2}\phi, \tag{4.14}$$

in view of (4.2) and (4.3). Similarly, the relations

$$\Delta J_\mu(\phi, \alpha) = J_\mu(\phi, \alpha - 2), \qquad \mu = 1, 2, 3, 4 \tag{4.15}$$

hold. From (4.13) we see that

$$\Delta J_\mu(\xi, 2) = 0, \qquad \mu = 1, 2, 3, 4. \tag{4.16}$$

The relation (4.4) follows as in the scalar case from the formula (d) of §3 which may be written

$$(V^\alpha, V^\beta)_{D_y^x} = \int_{D_y^x} V^\alpha(x, z) \wedge {}*V^\beta(y, z) = V^{\alpha+\beta}(x, y), \tag{4.17}$$

and which we prove in essentially the same way. We apply Green's theorem to D_y^x for $R(\alpha) > m$, $R(\beta) > m$, and, noting that the surface terms vanish, we have

$$\begin{aligned}(V^{\alpha+2n}(x, z), V^\beta(y, z))_{D_y^x} &= (V^{\alpha+2n}(x, z), \Delta_z^n V^{\beta+2n}(y, z))_{D_y^x} \\ &= (\Delta_z^n V^{\alpha+2n}(x, z), V^{\beta+2n}(y, z))_{D_y^x} = (V^\alpha(x, z), V^{\beta+2n}(y, z))_{D_y^x},\end{aligned}$$

$n = 0, 1, 2, \ldots$, and, letting $\alpha, \beta \to 0$ we find [9, p. 196]

$$V^{2n}(x, y) = V^{2n}(y, x).$$

It follows that the coefficient forms V_k are symmetric, hence

$$V^\alpha(x, y) = V^\alpha(y, x), \tag{4.18}$$

so that condition (c) of §3 is satisfied.

Consider the expression

$$(V^\alpha, V^\beta)_{D_y^x} - V^{\alpha+\beta}; \qquad R(\alpha) > m, R(\beta) > m.$$

A typical component $g(\alpha)$ of this double form is an analytic function of α which satisfies the estimate

$$|g(\alpha)| < \left| C\left(\frac{\alpha}{e}\right)^{-\alpha} \alpha^\beta s^\alpha \right| \tag{4.19}$$

for $R(\alpha) > m$. Also $g(\alpha)$ is zero for $\alpha = 2n$ $(n = 0, 1, 2 \ldots)$ by the preceding formulae. We may now apply the following theorem [7, Abs. III, problem 298, pp. 142, 327]: If two analytic functions $g(z)$, $h(z)$ are regular for $R(z) > a$, and $h(z) \not\equiv 0$ for $R(z) > a$, and

$$|g(z)| < |h(z)|,$$

then, denoting by z_n the zeros of $g(z)$ with $R(z_n) > a$, $|z_n| > 1$, the divergence of the series

$$\sum_n R(1/z_n)$$

implies that $g(z)$ is identically zero. Taking $h(\alpha)$ equal to the expression in the absolute bars on the right-hand side of (4.19), and noting that the series for the zeros of $g(\alpha)$ is the divergent harmonic series, we conclude that $g(\alpha) \equiv 0$. Hence (4.17) holds for $R(\alpha) > m$, $R(\beta) > m$, and therefore in general.

Using (4.17) we find by inverting a certain double integral that

$$I^{\alpha} J_{\mu}(\phi, \beta) = J_{\mu}(\phi, \alpha + \beta), \qquad \mu = 1, 2, 3, 4; \tag{4.20}$$

for $R(\alpha)$, $R(\beta)$ sufficiently large, and therefore in general.

The kernel V^{α} has certain additional properties which follow from the fact that if an operator T commutes with Δ it commutes with I^{α}. We first establish this assertion. Let T be a (linear) operator such that $T\Delta = \Delta T$. We wish to prove that

$$TI^{\alpha}\phi = I^{\alpha}T\phi. \tag{4.21}$$

Now (4.21) is certainly true for $\alpha = 0$, by (4.2). Using (4.3), we can prove that it holds also for $\alpha = 2n$ $(n = 0, 1, 2, \ldots)$. For,

$$TI^{2}\phi = I^{2}\Delta TI^{2}\phi = I^{2}T\Delta I^{2}\phi = I^{2}T\phi,$$

since T and Δ commute, and Δ is inverse to I^{2}. By iteration of this, we find that (4.20) holds for $\alpha = 2n$. Consider now any typical component of

$$TI^{\alpha}\phi - I^{\alpha}T\phi, \tag{4.22}$$

this component is an analytic function of α which admits the estimate (4.19) and vanishes for $\alpha = 2n$. By the theorem we have just quoted above, we must have (4.22) identically zero. This proves that T commutes with I^{α}.

Now Δ commutes with $*$, d, and δ. Applying our remark to the $*$ operator, we have

$$\underset{x}{*}(\phi(y), V_p^{\alpha}(x, y))_{D^x} = (*\phi(y), V_{m-p}^{\alpha}(x, y))_{D^x} = (-)^{mp+p}(\phi(y), \underset{y}{*} V_{m-p}^{\alpha})_{D^y}$$

for all ϕ. Taking the dual with respect to x, we find

$$V_p^{\alpha}(x, y) = \underset{xy}{**} V_{m-p}^{\alpha}(x, y). \tag{4.23}$$

Similarly, from $dI^{\alpha} = I^{\alpha}d$ we find that

$$\underset{x}{d}\, V_p^{\alpha}(x, y) = \delta\, \underset{y}{} V_{p+1}^{\alpha}(x, y). \tag{4.24}$$

The relation $\delta I^\alpha = I^\alpha \delta$ yields nothing additional. Note that (4.23) and (4.24) are essentially the same as relations satisfied by de Rham's Green's form [8a] and for the same reason.

From these relations it follows that the "single layer" potentials J_1 and J_4 are dual in the sense that

$$J_1(*\phi, \alpha) = *J_4(\phi, \alpha). \tag{4.25}$$

Similarly, the "double layer" potentials J_2 and J_3 are related by the equation

$$J_2(*\phi, \alpha) = *J_3(\phi, \alpha). \tag{4.26}$$

Since derivatives may be calculated formally for the potentials J_μ, it follows from (4.8) and (4.24) that

$$\delta J_2(\phi, \alpha) = 0, \; dJ_3(\phi, \alpha) = 0. \tag{4.27}$$

In order to show that our solution (4.12) satisfies the conditions (4.11), we must examine the potentials I^2 and $J_\mu(\xi, 2)$ as x tends to the surface S. First let us consider the volume potential I^2. Just as in the scalar case it follows that $I^2\rho \to 0$ since the conoid D_s^x becomes infinitesimal as x tends to S. The first derivatives of $I^2\rho$ also tend to zero.

As for the surface potentials, it will be sufficient, in view of (4.25) and (4.26), to consider J_1 and J_2. We shall reduce these expressions to integrals of the type studied by Fremberg, and apply his results. We may assume that S has the Lorentz form (3.10); it follows that as x tends to S, s becomes small of the same order as ξ^m. We may discard all terms of V^α except the first, noting that these terms, together with their first derivatives, become negligible in comparison with the first term.

Considering first J_1, we have

$$J_1(\xi, \alpha) \sim -\int_{S'} \frac{s^{\alpha-m}}{H_m(\alpha)} I_p \wedge *d\xi,$$

whence, on S,

$$(J_1)_{i_1 \ldots i_p} = -\int_{S'} \frac{s^{\alpha-m}}{H_m(\alpha)} (I_{i_1 \ldots i_p} \wedge *d\xi)_{1 \ldots m-1} \, dx^1 \wedge \ldots \wedge dx^{m-1}.$$

In the integrand we have

$$(I_{i_1 \ldots i_p} \wedge *d\xi)_{1 \ldots (m-1)} = (-1)^p (d\xi)_{i_1 \ldots i_p m}.$$

Hence the components of tJ, are simple layer potentials of Fremberg's type, so that [4, p. 44],

$$tJ_1 = 0, \; \frac{d}{dn}(tJ_1)_{i_1 \ldots i_p} = (-1)^p (d\xi)_{i_1 \ldots i_p m}.$$

Here n denotes the inward normal to the surface of the conoid. The integrand is zero if, say, $i_p = m$, so that

$$nJ_1 = 0, \; \frac{d}{dn}(nJ_1) = 0.$$

We now calculate

$$\begin{aligned}(ndJ)_{i_1\dots i_p m} &= \Gamma_{i_1\dots i_p m}{}^{j(j_1\dots j_p)}\partial_j J_{1(j_1\dots j_p)}\\ &= \Gamma_{i_1\dots i_p m}{}^{m(j_1\dots j_p)}\partial_m J_{1(j_1\dots j_p)} \text{ since } nJ_1 = 0\\ &= (-1)^p \frac{\partial}{\partial x^m} J_{1(i_1\dots i_p)}\\ &= (d\xi)_{i_1\dots i_p m}.\end{aligned}$$

Finally, we have

$$(t\delta J_1)_{i_1\dots i_{p-1}} = -\Gamma_{i i_1\dots i_{p-1}}{}^{(j_1\dots j_p)} D^i J_{1(j_1\dots j_p)}.$$

If the summation index i is less than m, then none of the j_μ are equal to m, hence we get a tangential derivative of a component of tJ_1, which yields zero. If $i = m$, we get the normal derivative of a component of nJ_1, which is again zero. Altogether, we have

$$tJ_1 = 0,\ nJ_1 = 0,\ ndJ_1 = nd\xi,\ t\delta J_1 = 0. \tag{4.28}$$

A similar analysis holds for J_2. We find

$$\begin{aligned}J_2(\xi,\alpha) &\sim \int_{S^+} \xi \wedge *d\frac{s^{\alpha-m}}{H_m(\alpha)} I_p\\ &= \int_{S^+}\left(\xi \wedge *I_p\, d\frac{s^{\alpha-m}}{H_m(\alpha)}\right)_{1\dots(m-1)} dx^1 \dots dx^{m-1}.\end{aligned}$$

Taking a typical normal component of J_2, we find that the integrand is of the form

$$\xi_{i_1\dots i_{p-1} i}\frac{\partial}{\partial x^j}\frac{s^{\alpha-m}}{H_m(\alpha)}, \qquad i, i_1 \dots i_{p-1} < m,$$

which shows that components of nJ_2 are tangential derivatives of single layer potentials. As we have seen above, these are zero. Hence $nJ_2 = 0$. For tJ_2 we find

$$\left(\xi \wedge *\left\{I_{p\,i_1\dots i_p} \wedge d\frac{s^{\alpha-m}}{H_m(\alpha)}\right\}\right)_{1\dots(m-1)} = \xi_{i_1\dots i_p}\frac{d}{dn}\frac{s^{\alpha-m}}{H_m(\alpha)},$$

which leads to a double layer potential of Fremberg's type for $(tJ_2)_{i_1\dots i_p}$. From his results [4, p. 48] we conclude that

$$tJ_2 = t\xi,\ \frac{d}{dn}tJ_2 = 0.$$

We note from (4.27) that $t\delta J_2$ is zero. Again,

$$(dJ_2)_{i_1\dots i_p m} = \Gamma_{i_1\dots i_p m}{}^{j(j_1\dots j_p)}\partial_j J_{2(j_1\dots j_p)}.$$

For $j = m$, we obtain a normal derivative of a component of tJ_2, which is zero. For $j \neq m$ we obtain a tangential derivative of a component of nJ_2, which also yields zero. Collecting our results, we have

$$tJ_2 = t\xi, \quad nJ_2 = 0, \quad ndJ_2 = 0, \quad t\delta J_2 = 0. \tag{4.29}$$

The corresponding results for J_3 and J_4 follow from (4.25) and (4.26).

The solution (4.12) may now be verified. From (4.14) and (4.16) we have

$$\Delta\phi = \Delta\{I^2\rho + \sum_\mu J_\mu(\xi, 2)\} = \rho,$$

so that (4.10) holds. To verify (4.11) we note that the volume potential and its first derivatives contribute nothing on S. Applying (4.28), (4.29) and their duals, the result (4.11) follows.

The proof can be used to show that the kernel $V^\alpha(x, y)$ is uniquely determined independently of the coordinate system in which it was constructed. The volume potential $I^2\rho$ is uniquely determined, hence so is V^2. It follows from (4.17) that V^{2n} is unique for $n = 1, 2, 3, \ldots$; and the uniqueness theorem for analytic functions quoted above shows that V^α is unique for all values of α. The kernel V^α is defined in a small region only, but repetition of the integration process shows that the solution can be extended throughout R.

5. The harmonic field equations. We consider the system of first order equations for the components of the p-tensor $\phi_{i_1 \ldots i_p}$;

$$d\phi = \rho, \quad \delta\phi = \sigma \tag{5.1}$$

and the problem of solving this system under conditions sufficient to determine the solution uniquely. When ρ and σ are replaced by zero in (5.1) we have the equations of harmonic fields. The solution of our problem will utilize Theorem I.

Let S be a spacelike surface and R its region of dependence, as before. Obvious necessary conditions that (1) should possess a solution are that ρ be a derived form, and σ the dual of a derived form. Hence we impose the conditions

$$\begin{aligned} d\rho &= 0, \qquad \int_{z_{p+1}} \rho = 0, \\ \delta\sigma &= 0, \qquad \int_{z_{n-p+1}} *\sigma = 0 \end{aligned} \tag{5.2}$$

for all cycles z_{p+1}, z_{n-p+1} of R.

Since (5.1) is a first order system, it is to be expected that the assignment of $t\phi$ and $n\phi$ on S will determine the solution. That this is indeed true follows from Lemma II. For, the homogeneous system

$$d\phi = 0, \quad \delta\phi = 0 \text{ in } R, \quad \phi \equiv 0 \text{ on } S \tag{5.3}$$

has only $\phi = 0$ as solution. To show this, we observe that $\Delta\phi = 0$, $t\phi = 0$, $n\phi = 0$, $nd\phi = 0$ and $t\delta\phi = 0$. Hence $\phi = 0$ in R.

Returning to the system (5.1) we observe that if $\phi \equiv \xi$ on S, then ξ must satisfy the conditions

$$td\xi = t\rho, \quad n\delta\xi = n\sigma, \tag{5.4}$$

since the corresponding component equations contain only derivatives in

directions lying in S. A p-form ξ defined on S which satisfies (5.4) will be called admissible with respect to (5.1), or, more briefly, admissible.

As a simpler example of the method which we shall use to solve (5.1), we consider the homogeneous equations

$$d\phi = 0, \quad \delta\phi = 0 \tag{5.5}$$

with the boundary condition $\phi \equiv \xi \in C\,[\frac{1}{2}(m+1)]$ on S, where ξ is admissible with respect to (5.5). Let ϕ be the solution of $\Delta\phi = 0$, such that $t\phi = t\xi$, $n\phi = n\xi$, $nd\phi = 0$, $n\delta\phi = 0$ on S. Thus

$$\phi = J_2(\xi, 2) + J_3(\xi, 2). \tag{5.6}$$

It remains to be shown that ϕ is closed and coclosed. But $td\phi = td\xi = 0$, $nd\phi = 0$, $\Delta(d\phi) = d\,\Delta\phi = 0$, and $nd(d\phi) = 0$ since $d\,.\,d\phi$ is zero. Finally, we have

$$t\delta(d\phi) = -\,td\delta\phi = -\,\underset{s}{d}t\delta\phi = 0,$$

since $\delta\phi = 0$ on S. Hence, by Lemma II, $d\phi \equiv 0$. Reasoning exactly dual to the preceding shows that $\delta\phi = 0$ in R. Hence ϕ is the unique harmonic field which assumes the admissible value ξ on S.

Since the equations (5.1) are non-homogeneous, we shall need to treat them in a less direct fashion. We divide the problem into two parts. Consider first the problem of solving the system

$$d\phi = \rho, \quad \delta\phi = 0 \tag{5.7}$$

where

$$t\phi = t\xi, \quad n\phi = 0 \tag{5.8}$$

on S, and $t\xi$ is admissible with respect to (5.7). That is,

$$td\xi = t\rho, \tag{5.9}$$

on S.

Assume for the moment that a solution ϕ exists. Then we have from (5.7),

$$d*\phi = 0,$$

and

$$\int_{z_{n-p}} *\phi = \int_{z'^s{}_{n-p}} *\phi = 0$$

from (5.8), where z_{n-p} is any (absolute) cycle of R, and $z'^s{}_{n-p}$ is a homologous cycle of S. The cycle $z'^s{}_{n-p}$ homologous to z_{n-p} always exists (see §1). It follows that $*\phi$ is a derived form in R:

$$\phi = \delta X. \tag{5.10}$$

We now obtain

$$d\delta X = \rho,$$

which suggests that we attempt to solve (5.7) by solving

$$\Delta X = -\rho \tag{5.11}$$

under conditions which ensure that $d\mathrm{X} \equiv 0$. This is a kind of "gauge" condition. It is not difficult to write down the auxiliary conditions which X must satisfy on S. They are

$$t\mathrm{X} = 0,\ n\mathrm{X} = 0,\ nd\mathrm{X} = 0,\ t\delta\mathrm{X} = t\phi = t\xi. \tag{5.12}$$

These conditions ensure that (5.8) are satisfied. To prove that X is closed, note that

$$\Delta(d\mathrm{X}) = d(\Delta\mathrm{X}) = -d\rho = 0,$$

that $td\mathrm{X} = 0$, $nd\mathrm{X} = 0$, and that $nd\,.\,d\mathrm{X}$ is zero. Finally, we have

$$\begin{aligned} t\delta d\mathrm{X} &= -td\delta\mathrm{X} + t\rho = -td\delta\mathrm{X} + td\xi \\ &= td(\xi - \delta\mathrm{X}) = \underset{s}{d}\, t(\xi - \delta\mathrm{X}) = 0 \end{aligned}$$

in view of (5.9) and (5.12). Hence $d\mathrm{X} \equiv 0$ in R.

From Theorem I we have

$$\mathrm{X} = -I^2\rho + \int_{S^*} \xi \wedge {}^*V_{p+1}^{m+2}\Big|_{u=0},$$

whence from (5.10)

$$\begin{aligned} \phi &= -\delta I^2\rho + \delta \int_{S^*} \xi \wedge {}^*V_{p+1}^{m+2}\Big|_{u=0} \\ &= -I^2\delta\rho + J_2(\xi, 2), \end{aligned} \tag{5.13}$$

in view of (4.8), (4.24) and the permissibility of differentiating under the integral sign.

The dual problem, in which

$$d\phi = 0,\ \delta\phi = \sigma, \tag{5.14}$$

and the boundary conditions are easily written down, can be solved in the same way, or by use of (4.23) and (4.26). The solution is

$$\phi = -I^2 d\sigma + J_3(\xi, 2). \tag{5.15}$$

Summing up these results, we have

THEOREM II. *Let S be a spacelike surface of class $C[\frac{1}{2}(m+3)]$, and ρ, σ forms of class $C[\frac{1}{2}(m+3)]$, of degrees $(p+1)$ and $(p-1)$ respectively, which satisfy* (5.2). *Let $\xi \in C[\frac{1}{2}(m+1)]$ be a p-form admissible with respect to* (5.1). *Then there exists a unique solution of* (5.1) *defined in R, given by*

$$\phi = -I^2(\delta\rho + d\sigma) + J_2(\xi, 2) + J_3(\xi, 2). \tag{5.16}$$

We note that if ρ and σ are harmonic fields, the first term of (5.16) vanishes, and the solution is formally the same as (5.6). However ξ must satisfy the conditions of admissibility (5.4).

6. Concluding remarks. We consider certain special cases of the two theorems which result when further restrictions are imposed on the manifold M or

the metric. A comparison of the present results with the boundary value theorems for p-tensors in the elliptic case will also be made.

Under certain circumstances the solutions discussed in Theorems I and II may be valid in a wider region than has been described. To begin with, we have restricted ourselves to one "side" of the initial surface and it is clear that the solution is valid both for $t > 0$ and $t < 0$ in the appropriate regions R^+, R^-. The solution is also determined in the region R_1 associated with any surface S_1 which is spacelike and lies in R. Suppose for example that M is the product of an $(m-1)$-dimensional closed manifold and a line $(-\infty < t < +\infty)$; with metric

$$ds^2 = dt^2 - g_{\alpha\beta}(t)\, dx^\alpha\, dx^\beta$$

where $g_{\alpha\beta}(t)$ gives rise to a positive definite line element depending on t as a parameter. Let S be the surface $t = 0$: the solution is then determined throughout the entire manifold. If the $g_{\alpha\beta}(t)$ are independent of t, the B.L. equation is separable:

$$\phi_{tt} = \Delta_s \phi$$

where Δ_s is the B.L. operator for S. Setting $\phi = e^{\lambda t} u$, we would have

$$\Delta_s u + \lambda^2 u = 0$$

just as in the scalar case.

If the metric is of certain restricted types, the B.L. equation will separate into $\binom{m}{p}$ independent component equations. This is the case for the Lorentz metric

$$ds^2 = dt^2 - (dx^1)^2 - \ldots - (dx^{m-1})^2.$$

Each component of ϕ then satisfies a wave equation

$$\phi_{tt} = \sum_\alpha \phi_{x^\alpha x^\alpha}.$$

The Riesz p-tensor kernel reduces to the first term of the series:

$$V_p^m(x, y) = \frac{s^{\alpha-m}}{H_m(\alpha)} \sum_{i_1<\ldots<i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \cdot dy^{i_1} \wedge \ldots \wedge dy^{i_p},$$

for which the relations (4.23) and (4.24) are easily verified. This case can therefore be treated by scalar methods.

Of particular interest is the case $m = 4$, $p = 2$, since the equations

$$d\phi = \rho,\ \delta\phi = 0$$

are then, in the Lorentz metric, equivalent to Maxwell's equations, with $\mu = \epsilon = 1$. In a general metric it would be natural to retain this invariant formulation. The two-tensor $\phi_{i_1 i_2}$ is the electromagnetic field tensor $F_{i_1 i_2}$ and the dual of the three-form ρ is the charge-current vector. Theorem II may therefore be interpreted as a characterization of electromagnetic fields in a general relativity

metric. The conservation of charge current is expressed by the fact that ρ is a derived form, in other words that

$$\int_{z_n} \rho = 0$$

for all cycles z_n.

It is interesting to compare our results with the boundary value theorems for p-tensors in the elliptic case, and with the classical theory of differential equations ($p = 0$ or $p = m$). We consider first the B.L. equation. The result in the elliptic case is that the solution exists and is unique if $t\phi$ and $n\phi$ are given on the boundary. This result has, however, only been proved under certain topological restrictions [3]. In the hyperbolic case the initial surface does not bound the domain of determination of the solution, but the additional data $nd\phi$ and $t\delta\phi$ are needed to determine the solution uniquely. If $p = 0$, $n\phi$ is zero automatically, and so is $t\delta\phi$. Hence the data reduce to $t\phi = \phi$ and

$$nd\phi = \frac{\partial \phi}{\partial n}\, dn,$$

in the hyperbolic case, and to ϕ in the elliptic case. A similar remark holds if $p = m$.

For the harmonic field equations the elliptic boundary value problem is uniquely solvable given either the normal boundary component and the absolute periods, or the tangential boundary components together with the relative periods. In the hyperbolic case both $t\phi$ and $n\phi$ are to be prescribed on the (smaller) initial surface. The region R in which the hyperbolic solutions are defined has no relative cycles (modulo the initial surface) and the periods on all absolute cycles are fixed by the given data $t\phi$ and $n\phi$ on S since all absolute cycles of R are homologous to cycles of S. In the hyperbolic problem, therefore, period conditions are not needed explicitly.

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Massachusetts Institute of Technology and
University of Toronto

PARTITIONING AN ARITHMETIC INTERVAL

WILLIAM GUSTIN

The purpose of this paper is to characterize all ways in which an initial interval of natural numbers can be partitioned into a unique arithmetic sum of certain of its subsets.

Preliminaries. We set forth below certain explanations, conventions, and definitions pertinent to our subject.

Numbers and sets. By number we mean a natural (that is to say, an arithmetic or finite ordinal) number 0, 1, 2, ... Every set of numbers is required to contain 0. A set containing a number other than 0 will be called proper. The improper set comprising solely 0 will be written 0. The set consisting of all numbers dx as x ranges over a set X is denoted by dX.

Intervals. An (initial arithmetic) interval is a set of numbers containing every predecessor of each number in it. A finite interval containing d numbers thus consists of the first d numbers $r < d$ where $d \geqslant 1$. An infinite interval contains all natural numbers, that is, all numbers $r < \omega$ where ω is the first transfinite ordinal. Denote by $I(d)$ the interval consisting of all numbers $r < d$ where $1 \leqslant d \leqslant \omega$. $I(d)$ is finite or infinite according as $d < \omega$ or $d = \omega$, proper or improper according as $d > 1$ or $d = 1$. For this reason we call an ordinal d proper if $1 < d \leqslant \omega$.

Proper sequence. A finite (terminating) or infinite (non-terminating) numerically indexed[1] sequence $d_1, d_2, d_3, \ldots$ of ordinals will be called proper if not only is each ordinal in it proper but its ordinal product is also. Thus $1 < d_m \leqslant \omega$ for each index m and $1 < d_1 d_2 d_3 \ldots \leqslant \omega$. Consequently, if there is an index m such that $d_m = \omega$, the sequence terminates at this index m.

Partitions. We shall say that a set X is partitioned into (a unique arithmetic sum of) finitely or infinitely many numerically indexed[2] sets X^μ if every number x in X can be uniquely expressed in the form $x = \sum x^\mu$ where the unique coordinate x^μ of x belongs to X^μ for each index μ; and if furthermore every number x of this form belongs to X. To indicate this we write $X = \sum X^\mu$. It is easily seen that order and grouping of the terms X^μ in such a sum are not significant and that any two differently indexed terms X^μ have 0, and only 0, in common.

Division algorithm. As an apposite example of partitioning an interval consider the division algorithm. Let d_1, d_2 be a proper sequence of two ordinals. According to the division algorithm every number $r < d_1 d_2$ can be uniquely expressed in the form

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[1]Subscript indices indicate that the order of indexing is significant.

[2]Superscript indices indicate that the order of indexing is not significant.

$$r = r_1 + d_1 r_2,$$

where $r_1 < d_1$ and $r_2 < d_2$; and furthermore every number r of this form is $< d_1 d_2$. That is,

$$I(d_1 d_2) = I(d_1) + d_1 I(d_2).$$

Statement of results. Our first result concerning partitions of an interval merely extends the division algorithm to any proper sequence and provides a recipe for constructing interval partitions. Our second result asserts that every interval partition can be constructed, and in a certain sense uniquely, by following this recipe.

THEOREM I. *Let $d_1, d_2, d_3, \ldots$ be a proper sequence of ordinals. Then*

$$I(d_1 d_2 d_3 \ldots) = I(d_1) + d_1 I(d_2) + d_1 d_2 I(d_3) + \ldots.$$

Coarser repartitions of $I(d_1 d_2 d_3 \ldots)$ can be formed from this partition by grouping the above terms together in arbitrary fashion.

THEOREM II. *Let $I = \sum X^\alpha$ be a partition of a proper interval I by indexed sets X^α. There then exists a unique proper sequence of ordinals $d_1, d_2, d_3, \ldots$ generating the following finer repartition of I:*

$$I = I(d_1 d_2 d_3 \ldots) = I(d_1) + d_1 I(d_2) + d_1 d_2 I(d_3) + \ldots,$$

the sets X^α of the given partition of I being formed by appropriately grouping together the terms of this finer repartition of I, with consecutive terms allocated to different X^α sets.

Proofs. We shall establish the results stated above somewhat formally by using three lemmas. The first lemma is used in proving the first theorem, all three lemmas in proving the second.

LEMMA 1. *Let there be given a proper sequence of ordinals d_m, and sequences of sets A_m and X_m, all sequences of the same length, such that:*

$$X_m = \begin{cases} A_m & (m \text{ terminal}), \\ A_m + d_m X_{m+1} & (m \text{ non-terminal}). \end{cases}$$

Then

$$X_1 = A_1 + d_1 A_2 + d_1 d_2 A_3 + \ldots.$$

Proof. For each x_1 in X_1 the sequences x_m in X_m, a_m in A_m are uniquely determined by recursion as follows:

$$x_m = \begin{cases} a_m & (m \text{ terminal}), \\ a_m + d_m x_{m+1} & (m \text{ non-terminal}), \end{cases}$$

so for each m,

$$x_1 = a_1 + d_1 a_2 + \ldots + d_1 d_2 \ldots d_{m-1} x_m.$$

If m is terminal $x_m = a_m$. If, on the other hand, the sequence d_m does not terminate, then by choosing m so large that $d_1 d_2 \ldots d_m > x_1$ we find $x_n = 0$ and $a_n = 0$ for all $n > m$. Therefore, whether the sequences terminate or not,

$$x_1 = a_1 + d_1 a_2 + d_1 d_2 a_3 + \ldots,$$

this representation being unique, as was to be shown.

Proof of Theorem I. Let d_m be a proper sequence of ordinals, terminating or not. According to the division algorithm

$$I(d_m d_{m+1} \ldots) = I(d_m) + d_m I(d_{m+1} \ldots)$$

for all nonterminal m; so the hypotheses of Lemma 1 are satisfied by taking $A_m = I(d_m)$ and $X_m = I(d_m d_{m+1} \ldots)$, whereupon

$$I(d_1 d_2 d_3 \ldots) = I(d_1) + d_1 I(d_2) + d_1 d_2 I(d_3) + \ldots,$$

as was to be shown.

LEMMA 2. *Let $I = A + B$ be a partition of a proper interval I and d a proper ordinal such that A contains $I(d)$ but not d. There then exist: an interval $\bar{I}$ with $\bar{I} = 0$ if $d = \omega$, and sets $\bar{A}$ and $\bar{B}$ such that*

$$A = I(d) + d\bar{A}, \tag{1}$$

$$B = \quad d\bar{B}, \tag{2}$$

$$\bar{I} = \bar{A} + \bar{B}. \tag{3}$$

Proof. If $d = \omega$ it is evident that $A = I = I(\omega)$ and $B = 0$, so the lemma is verified by taking $\bar{I}$, $\bar{A}$, $\bar{B}$ all equal to 0. Furthermore, propositions (1) and (2) imply the remainder of the lemma. For since $A + B$ is a unique sum, $\bar{A} + \bar{B}$ is also a unique sum. Define $\bar{I} = \bar{A} + \bar{B}$. Thus $\bar{I}$ satisfies (3) and in addition

$$I = I(d) + d\bar{I}.$$

From this it follows that $\bar{I}$ is an interval. For if $u < v$ with v in $\bar{I}$, then $du < dv$ with dv in I, whence du is in I and hence u is in $\bar{I}$.

It therefore remains to prove (1) and (2) for $d < \omega$. Now (1) and (2) together are equivalent to affirming for each $q = 0, 1, 2, \ldots$ the following proposition.

$\mathfrak{P}_q$: *if $r < d$ and $r + dq$ belongs to A or B, then $\rho + dq$ is in A for all $\rho < d$ in case $r + dq$ is in A, and $r = 0$ in case $r + dq$ is in B.*

We establish these propositions $\mathfrak{P}_q$ by induction. Obviously $\mathfrak{P}_0$ is true. Assume then that $\mathfrak{P}_n$ holds for all $n < q$ with $q > 1$: to prove $\mathfrak{P}_q$. We do this by first establishing from the induction hypothesis the following weaker proposition $\mathfrak{p}_q$ obtained from $\mathfrak{P}_q$ by putting $r = 0$:

$\mathfrak{p}_q$: *$\rho + dq$ is in A for all $\rho < d$ in case dq is in A.*

Then from $\mathfrak{p}_q$ and the induction hypothesis we prove $\mathfrak{P}_q$.

To prove $\mathfrak{p}_q$, let dq be in A: we are to show that $\rho + dq$ is in A for all $\rho < d$. Because $q > 1$, $d < dq$ and dq is in I, so d is in I. Since A contains $I(d)$ but not d, the B-coordinate of d is > 0 and hence $> d$, so must equal d. Therefore d is in B. This, together with dq in A, shows that $dq + d$ belongs to $A + B = I$.

Consequently $\rho + dq$, being $< dq + d$, is also in I. Let $\rho + dq$ have coordinates a in A, b in B. We wish to show that $\rho + dq$ is in A, or, what amounts to the same, that $b = 0$. Suppose, to the contrary, that $b > 0$. Write b in the form $b = r + d\beta$ with $r < d$. Hence $\beta \leqslant q$. If $\beta = q$, then $r > 0$, for otherwise we would have $dq = b$ in B as well as dq in A. The number $dq + d$ with coordinates dq in A, d in B thus has alternative coordinates $d - r$ in A, $r + dq = b$ in B; which is impossible. On the other hand, if $\beta < q$, then $r = 0$ by the induction hypothesis, so $b = d\beta$. Thus a has the form $a = \rho + d\alpha$ where $\alpha + \beta = q$ and $0 < \alpha < q$ since $0 < \beta < q$. Therefore $d\alpha$ is in A by the induction hypothesis: so dq in A has positive coordinates $d\alpha$ in A, $d\beta$ in B; which is impossible. This dilemma proves our contention that $\rho + dq$ is in A. Thus $\mathfrak{p}_q$ is proved.

To prove $\mathfrak{P}_q$, let $r < d$ and $r + dq$ belong to A or B: we are to show that $\rho + dq$ is in A for all $\rho < d$ in case $r + dq$ is in A, and that $r = 0$ in case $r + dq$ is in B. Now $r + dq$ is in I, so dq is in I also. Let dq have coordinates a in A, b in B. Then b has the form $b = d\beta$ with $\beta \leqslant q$: obviously in case $b = dq$, by the induction hypothesis in case $b < dq$. Hence a has the form $a = d\alpha$ with $\alpha \leqslant q$. Consequently $r + d\alpha$ is in A: by $\mathfrak{p}_q$ in case $\alpha = q$, by the induction hypothesis in case $\alpha < q$. The number $r + dq$ then has the coordinates $r + d\alpha$ in A, $d\beta$ in B. Therefore if $r + dq$ is in A, $d\beta = 0$ so $dq = d\alpha$ is in A; whence $\rho + dq$ is in A for all $\rho < d$ by $\mathfrak{p}_q$. And if $r + dq$ is in B, $r + d\alpha = 0$; whence $r = 0$. This completes the proof of $\mathfrak{P}_q$ and hence, by induction, the proof of the lemma.

LEMMA 3. *Let $I = \sum X^\alpha$ be a partition of a proper interval I. There then exist: a proper ordinal d, an interval $\bar{I}$ with $\bar{I} = 0$ if $d = \omega$, an index α with d not in X^α, and sets $\bar{X}^\mu$, all of these unique, such that*

$$X^\mu = \delta^{\mu\alpha} I(d) + d\bar{X}^\mu, \tag{4}$$

$$\bar{I} = \sum \bar{X}^\mu, \tag{5}$$

where δ is Kronecker's delta: $\delta^{\mu\alpha} = 1$ or 0 according as $\mu = \alpha$ or not.

Proof. Since $I \neq 0, 1$ is in I. Therefore $1 = \sum x^\alpha$ with x^α in X^α. Clearly a unique index α exists such that $x^\alpha = 1$ and $x^\beta = 0$ for all remaining indices $\beta \neq \alpha$. Define $A = X^\alpha$ and $B = \sum X^\beta$. Then $I = A + B$. Let d be the smallest ordinal not in $A = X^\alpha$. Since 1 is in A, d is a proper ordinal. Now A contains $I(d)$ but not d, so the hypothesis of Lemma 2 is satisfied, and thus the conclusion, whereby the unique sets $\bar{I}, \bar{A}, \bar{B}$ are furnished. Therefore $\bar{I} = 0$ if $d = \omega$. Define $\bar{X}^\alpha = \bar{A}$; then by (1),

$$X^\alpha = I(d) + d\bar{X}^\alpha.$$

By (2), d divides every number in B and hence every number in each X^β, so X^β is of the form

$$X^\beta = d\bar{X}^\beta,$$

with $\bar{B} = \sum X^\beta$. Evidently (4) summarizes the two formulae above. As for (5),

$$\bar{I} = \bar{A} + \bar{B} = \bar{X}^\alpha + \sum \bar{X}^\beta = \sum \bar{X}^\mu$$

by virtue of (3). This completes proof of the lemma.

Proof of Theorem II. Let $I = \sum X^\mu$ be a partition of a proper interval I. By recursive use of Lemma 3 we construct: a proper sequence of ordinals d_m, a sequence of indices α_m, consecutive ones being different, and sequences of sets X_m^μ with $X_1^\mu = X^\mu$, all these sequences unique and of the same length, such that

$$X_m^\mu = \begin{cases} \delta^{\alpha_m \mu} I(d_m) & (m \text{ terminal}), \\ \delta^{\alpha_m \mu} I(d_m) + d_m X_{m+1}^\mu & (m \text{ non-terminal}). \end{cases}$$

Therefore, by Lemma 1,

$$X^\mu = \delta^{\alpha_1 \mu} I(d_1) + \delta^{\alpha_2 \mu} d_1 I(d_2) + \delta^{\alpha_3 \mu} d_1 d_2 I(d_3) + \ldots,$$

whereupon

$$I = \sum X^\mu = I(d_1) + d_1 I(d_2) + d_1 d_2 I(d_3) + \ldots,$$

as was to be shown.

This finer repartition of the given partition of I has evidently been constructed in a unique fashion; it will be called the resolution of the given partition. Any two consecutive terms of this resolution lie in different X^μ sets, or, as we shall say, are separated.

Number of partitions. Consider for given positive numbers m and n a partition of the interval $I(n)$ into m sets:

$$I(n) = X_1 + \ldots + X_m.$$

Call the ordered sequence of sets $X_1, \ldots, X_m$ an ordered m-partition of $I(n)$, and let $p_m(n)$ be the number of such partitions.

The values of this partition counting function[3] can be obtained recursively as follows. Let $n > 1$. The (terminating) resolution of an ordered m-partition of $I(n)$ may be formed from the resolution of an ordered m-partition of $I(d)$, where $d < n$ is a divisor of n, by adding the term $T = dI(n/d)$ as separated terminal term to one of the partitioning sets of $I(d)$. If $d = 1$, all m partitioning sets of $I(d)$ are 0, so T can be added to any one of these m sets. If $d > 1$, T can be added to any one of the m partitioning sets of $I(d)$ except that one which contains the terminal term of the resolution of the partition of $I(d)$. This procedure uniquely delivers all ordered m-partitions of $I(n)$, so

$$p_m(n) = 1 + (m - 1) \sum p_m(d),$$

the summation extending over all divisors $d < n$ of n. Though derived for $n > 1$, this formula is also valid for $n = 1$, since obviously $p_m(1) = 1$.

Institute for Advanced Study
Princeton, N.J.

[3] Two explicit formulae for this function are derived from the recursion relation developed here in Proc. Amer. Math. Soc., *3* (1952), 31–35.

PROOF OF A CONJECTURE OF SCHOENBERG ON THE GENERATING FUNCTION OF A TOTALLY POSITIVE SEQUENCE

ALBERT EDREI

Let

$$a_0, a_1, a_2, \ldots \tag{1}$$

be a sequence of real terms with which we associate the generating power series

$$a_0 + a_1 z + a_2 z^2 + \ldots = f(z). \tag{2}$$

We consider the following definition due to Schoenberg [7, p. 362]:

DEFINITION 1. *The sequence* (1) *is said to be totally positive if the infinite matrix*

$$\begin{matrix} a_0 & 0 & 0 & 0 & \ldots \\ a_1 & a_0 & 0 & 0 & \ldots \\ a_2 & a_1 & a_0 & 0 & \ldots \\ a_3 & a_2 & a_1 & a_0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot \end{matrix} \tag{3}$$

has only non-negative minors (of all orders, with any choice of rows and columns).

Schoenberg [7] proved that the coefficients of the series (2) form a totally positive sequence if $f(z)$ is a function of the form

$$f(z) = Cz^{\lambda} e^{\gamma z} \prod_{\nu=1}^{\infty} (1 + \alpha_\nu z) \Big/ \prod_{\nu=1}^{\infty} (1 - \beta_\nu z), \tag{4}$$

where λ is a non-negative integer and $C > 0$, $\gamma \geqslant 0$, $\alpha_\nu > 0$, $\beta_\nu > 0$, $\sum(\alpha_\nu + \beta_\nu) < +\infty$.

From now on, we shall assume $a_0 \neq 0$ (i.e., $\lambda = 0$); obviously this is only an apparent restriction.

Schoenberg [7, p. 367] also conjectured that functions of the form (4) are the only ones which generate totally positive sequences.

At the International Congress of Mathematicians [Cambridge, Massachusetts, 1950], Aissen, Schoenberg, and Whitney announced that they had proved the following proposition:

A. *If the coefficients of the series* (2) *form a totally positive sequence, then the series* (2) *is necessarily the expansion of a function of the form*

$$f(z) = e^{\phi(z)} \prod_{\nu=1}^{\infty} (1 + \alpha_\nu z) \Big/ \prod_{\nu=1}^{\infty} (1 - \beta_\nu z),$$

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where $\phi(z)$ is an integral function and $\alpha_\nu > 0$, $\beta_\nu > 0$, $\sum(\alpha_\nu + \beta_\nu) < +\infty$.

In trying to prove this proposition, I was able to complete the solution of Schoenberg's problem by showing:

B. *Under the assumptions of* A, $\phi(z)$ *necessarily reduces to a polynomial of the form* $\gamma z + \delta$ $(\gamma > 0)$.

A detailed account of the results of Aissen, Schoenberg, Whitney, and the author will be given in the Journal d'Analyse Mathématique [1; 3; 8]. These results yield a proof of Schoenberg's conjecture which differs substantially from my first approach to the subject. My excuse for presenting here this early arrangement is that my method essentially reduces to the construction of the Padé table of the generating function of a totally positive sequence. It discloses a number of surprisingly simple properties of such a table. Their investigation has been continued by R. J. Arms and the author and has led to theorems on the convergence of the Padé table and continued fractions associated with a totally positive sequence.

1. Definition and immediate consequences of the property (P).

DEFINITION 2. *Consider the determinants*

$$A_m^{(n)} = \begin{vmatrix} a_m & a_{m-1} & a_{m-2} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+2} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+3} \\ \cdots & & & \cdots & \\ a_{m+n-1} & a_{m+n-2} & a_{m+n-3} & \cdots & a_m \end{vmatrix}, \quad A_m^{(0)} = 1 \qquad (m > 0, n > 0)$$

where $a_\mu = 0$ for $\mu < 0$. We say that the power series (2) *has the property* (P) *if $A_m^{(n)} > 0$, for all $m > 0$ and all $n > 0$.*

The inequalities

$$a_m > 0, \qquad A_m^{(2)} > 0 \qquad (m = 0, 1, 2, \ldots),$$

imply

$$\frac{1}{R} = \lim_{m\to\infty} \frac{a_{m+1}}{a_m} < \frac{a_1}{a_0},$$

so that, if (2) has the property (P), its radius of convergence R, is positive.[1] It has been proved by Schoenberg [6, p. 558] that, *if $f(z)$ has the property* (P), (1) *is totally positive.* On the other hand, it is obvious from the definitions that the converse cannot be true.[2]

[1] By a similar argument [7, p. 362[, it is easily verified that the generating series of a totally positive sequence has a positive radius of convergence.

[2] The connection between the property (P) and total positivity is completely clarified by an unpublished result of R. J. Arms. Mr. Arms has shown that, if $f(z)$ denotes the generating function of a totally positive sequence, and if $f(z)$ does not have the property (P), then $f(z)$ reduces to a rational function.

Two observations, of purely formal character play an important part in this investigation.

I. *If $f(z)$ has the property* (P), *then*

$$\frac{1}{f(-z)} = b_0 + b_1 z + b_2 z^2 + \dots,$$

also has the property (P).

This is an immediate consequence of the elementary identities

$$a_0^{-n} A_m^{(n)} = b_0^{-m} B_n^{(m)} \qquad (m > 0,\ n > 0),$$

where $B_n^{(m)}$ is the determinant obtained by replacing, in the expression for $A_n^{(m)}$, the a's by the b's.

II. *Let $f(z)$ and $g(z)$ denote generating functions of totally positive sequences. Then the sequence $\{c_n\}$ defined by*

$$f(z)\, g(z) = c_0 + c_1 z + c_2 z^2 + \dots \qquad (f(0)\, g(0) \neq 0)$$

is totally positive. Moreover $f(z)g(z)$ has the property (P) *if at least one of the two factors f and g has this property.*

It is easily verified, by direct computation, that $e^{\varepsilon z}$ has the property (P). Hence the function $e^{\varepsilon z} f(z)$ has the property (P) as soon as (1) is totally positive and $\varepsilon > 0$ $(a_0 \neq 0)$.

2. Poles and zeros of $f(z)$. Let $f(z)$ have the property (P). Consider the polynomials $Q^{(m,n)}(z)$ defined by

$$Q^{(m,n)}(z) = \frac{1}{A_m^{(n)}} \begin{vmatrix} 1 & z & z^2 & \dots & z^n \\ a_{m+1} & a_m & a_{m-1} & \dots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \dots & a_{m-n+2} \\ & \dots & & \dots & \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \dots & a_m \end{vmatrix} \qquad (m > 0,\ n > 0),$$

$$Q^{(m,0)}(z) \equiv 1 \qquad (a_\mu = 0 \text{ for } \mu < 0),$$

$$S^{(m,n)}(z) = Q^{(m,n)}(-z) \qquad (m > 0,\ n > 0).$$

We prove the following seven assertions:

(i) *the coefficients of $S^{(m,n)}$ are all positive*;

(ii) $$\lim_{m \to \infty} Q^{(m,n)}(z) = Q^{(\infty,n)}(z) \text{ exists for all } n > 0;$$

(iii) $$\lim_{m \to \infty} \frac{A_{m+1}^{(n+1)} \big/ A_m^{(n+1)}}{A_{m+1}^{(n)} \big/ A_m^{(n)}} = \beta_{n+1} > 0 \text{ exists for all } n (\geqq 0);$$

(iv) *the polynomials $Q^{(\infty,n)}(z)$ are connected by the recurrence relations*

$$Q^{(m,n+1)}(z) = (1 - \beta_{n+1}z)\, Q^{(m,n)}(z) \qquad (n > 0);$$

(v) $\sum_{n=1}^{\infty} \beta_n$ *converges so that*

$$Q(z) = \lim_{n\to\infty} Q^{(m,n)}(z) = \prod_{n=1}^{\infty} (1 - \beta_n z)$$

is either a polynomial or an integral function of genus zero;

(vi) $$f(z)\, Q^{(m,n)}(z) = p_0^{(n)} + p_1^{(n)}z + p_2^{(n)}z^2 + \dots$$

converges for $|z| < 1/\beta_{n+1}$, *the coefficients* $p_\nu^{(n)}$ *are all positive;*

(vii) $$f(z)\, Q(z) = p_0 + p_1 z + p_2 z^2 + \dots$$

is an integral function, the coefficients p_ν *are non-negative.*

Proof. All the above assertions are easily deduced from the identities[2]

$$S^{(m,n+1)}(z) = S^{(m+1,n)}(z) + zS^{(m,n)}(z)\frac{A_{m+1}^{(n+1)} \big/ A_m^{(n+1)}}{A_{m+1}^{(n)} \big/ A_m^{(n)}}, \tag{5}$$

$$S^{(m+1,n+1)}(z) - S^{(m,n+1)}(z) = -zS^{(m,n)}(z)\frac{A_{m+1}^{(n+2)} \big/ A_m^{(n+1)}}{A_{m+1}^{(n+1)} \big/ A_m^{(n)}} = -z\gamma_{mn}S^{(m,n)}(z). \tag{6}$$

Assertion (i) is a consequence of a theorem of Schoenberg [6, p. 558]; it is also easy to deduce directly from (5). We now prove (ii).

From (6), we obtain

$$S^{(l+1,n+1)}(z) = S^{(m,n+1)}(z) - z\sum_{\mu=m}^{l} \gamma_{\mu n}S^{(\mu,n)}(z) \qquad (\gamma_{\mu n} > 0). \tag{7}$$

As the coefficients of the S's are positive, it follows from (7) that all the coefficients (except the constant term, which is 1) decrease if we increase l and keep n fixed. This proves (ii) and also the convergence of

$$\sum_{\mu=0}^{\infty} \gamma_{\mu n}.$$

Observing that, for a positive z,

$$\lim_{m\to\infty} zS^{(m,n)}(z) > 0,$$

we deduce from (5) the existence of the limit in (iii). This proves (iii) and (iv) simultaneously.

Consider (7) for $l = \infty$, $m = 0$, and compare the coefficients of z; this yields

$$\beta_1 + \beta_2 + \dots + \beta_{n+1} < a_1/a_0,$$

which proves (v).

To prove (vi), we consider the expansion

[2] These formulae are straightforward consequences of Jacobi's theorem on the minors of the adjoint of a given determinant.

$$f(z)Q^{(m,n)}(z) = \sum_{\mu=0}^{\infty} p_\mu^{(m,n)} z^\mu \tag{8}$$

and observe that the well-known identity [5, p. 430]

$$f(z)Q^{(m,n)}(z) = \frac{1}{A_m^{(n)}} \sum_{\mu=0}^{\infty} z^\mu \begin{vmatrix} a_\mu & a_{\mu-1} & a_{\mu-2} & \dots & a_{\mu-n} \\ a_{m+1} & a_m & a_{m-1} & \dots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \dots & a_{m-n+2} \\ & \dots & & \dots & \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \dots & a_m \end{vmatrix}, \tag{9}$$

gives us explicitly the coefficients $p_\mu^{(m,n)}$. Now the sequence of functions

$$f(z)\, Q^{(m,n)}(z) \qquad (m = 0, \dots, \infty) \tag{10}$$

is regular in the circle $|z| < R$, where R denotes the radius of convergence of (2).

For n fixed and $m \to \infty$, (10) converges uniformly to

$$f(z)\, Q^{(\infty,n)}(z) = p_0^{(n)} + p_1^{(n)} z^2 + \dots, \tag{11}$$

in any circle $|z| \leqslant R^* < R$.

From (7) we deduce

$$f(z)\, Q^{(\infty,n)}(z) = f(z)\, Q^{(m,n)}(z) + z \sum_{\mu=m}^{\infty} \gamma_{\mu,n-1} \{f(z)\, Q^{(\mu,n-1)}(z)\}. \tag{12}$$

Using (8), (11), (12) and Weierstrass's double-series theorem, we obtain

$$p_k^{(n)} = p_k^{(m,n)} + \sum_{\mu=m}^{\infty} \gamma_{\mu,n-1} p_{k-1}^{(\mu,n-1)} \qquad (k = 1, 2, 3, \dots). \tag{13}$$

The total positivity of (1) implies $p_k^{(m,n)} > 0$, when $k < m$, so that, (13) yields

$$p_k^{(n)} > p_k^{(m,n)} \qquad (k < m). \tag{14}$$

Observing that

$$p_{m+1}^{(m,n)} = 0,$$

and using (14), we also obtain

$$p_{m+1}^{(n)} < p_m^{(n-1)} (\gamma_{m,n-1} + \gamma_{m+1,n-1} + \gamma_{m+2,n-1} + \dots) \qquad (m > 0, n > 1). \tag{15}$$

Putting

$$\limsup_{m \to \infty} |(p_m^{(n)})^{1/m}| = \sigma_n \qquad (n = 0, 1, 2, \dots), \tag{16}$$

we deduce, from (15),

$$\sigma_n \leqslant \sigma_{n-1} \limsup_{m \to \infty} \left| \left(\sum_{\mu=m}^{\infty} \gamma_{\mu,n-1} \right)^{1/m} \right|. \tag{17}$$

Moreover, as all series of the form

$$\sum_{\mu=m}^{\infty} \gamma_{\mu n}$$

are convergent, (17) trivially implies

$$\sigma_n \leqslant \sigma_{n-1} \qquad (n = 1, 2, 3, \ldots). \tag{18}$$

We now verify the inequalities

$$\sigma_k \leqslant \beta_{k+1} \qquad (k = 0, 1, 2, \ldots), \tag{19}$$

by an induction over k.

We note that $p_\mu^{(0)} = a_\mu$, implies $\sigma_0 = \beta_1$, and assume that (19) is true for $k = 0, 1, 2, \ldots, n-1$. We distinguish two cases

(a) $$\beta_1\beta_2 \ldots \beta_n = 0,$$

(b) $$\beta_1\beta_2 \ldots \beta_n \neq 0.$$

The case (a) is immediately settled by observing that our induction assumption, and (18), then yield $\sigma_k = 0$, for $k > n$.

In the case (b), we observe that

$$\limsup_{m\to\infty} \left| \left(\sum_{\mu=m}^{\infty} \gamma_{\mu,n-1} \right)^{1/m} \right| < \frac{\beta_{n+1}}{\beta_n}, \tag{20}$$

is trivial if $\beta_{n+1} \geqslant \beta_n$.

If $\beta_{n+1} < \beta_n$, we return to the formulae (6) which define the quantities γ_{mn} as ratios of determinants.

Observing that (iii) implies[4]

$$\lim_{m\to\infty} \frac{A_{m+1}^{(n+1)}}{A_m^{(n+1)}} = \beta_1\beta_2 \ldots \beta_n\beta_{n+1},$$

$$\lim_{m\to\infty} \left| (A_m^{(n)})^{1/m} \right| = \beta_1\beta_2 \ldots \beta_n, \tag{21}$$

we obtain

$$\lim_{m\to\infty} (\gamma_{m,n-1})^{1/m} = \lim_{m\to\infty} \left(\frac{A_{m+1}^{(n+1)}}{A_{m+1}^{(n)}} \frac{A_m^{(n-1)}}{A_m^{(n)}} \right)^{1/m} = \frac{\beta_{n+1}}{\beta_n}.$$

We may then compare

$$\sum_{\mu=m}^{\infty} \gamma_{\mu,n-1}$$

to the remainder of a geometric series. This comparison leads again to (20). Combining (17) and (20), we complete the induction which proves (19). As (13) implies $p_\mu^{(n)} > 0$, assertion (vi) is also proved. Assertion (vii) follows from (vi) and Weierstrass's double-series theorem. Although the fact is not used in this paper, we note that (19) can be replaced by

$$\sigma_k = \beta_{k+1} \qquad (k = 0, 1, 2, \ldots). \tag{22}$$

[4]It should be noted that the validity of (21) is independent of the arguments following (14). Interrupting our proof after (14), and using (21), a direct application of Hadamard's theorem on polar singularities (see, for instance [2, p. 333])would have enabled us to verify that $f(z)$ is a meromorphic function. However, as the wording of Hadamard's theorem does not meet all our requirements, and as we do not need its more delicate parts, I have replaced its use by simple and elegant arguments which I owe to R. J. Arms.

To verify this, we assume that, for a definite value of $k(=n)$, $\sigma_n < \beta_{n+1}$. The point $1/\beta_{n+1}$ would then be interior to the circle of convergence of the expansion of $f(z)Q^{(\infty,n)}(z)$. The function $f(z)Q^{(\infty,n+1)}(z)$ would be regular and would vanish at $z = 1/\beta_{n+1}$. This is impossible in view of the positivity of the coefficients $p_\nu^{(n+1)}$. We finally observe that (18) and (22) imply

$$\beta_1 \geqslant \beta_2 \geqslant \beta_3 \geqslant \ldots .$$

3. Proof of Proposition A. The function $e^z f(z)$ certainly possesses the property (P); it is therefore sufficient to prove the proposition for functions which have this property.

By I and the seven assertions, there exist two integral functions $P(z)$ and $Q(z)$ such that

$$P(z) = \prod_{\nu=1}^{\infty} (1 - \alpha_\nu z) \qquad (\alpha_\nu > 0, \sum \alpha_\nu < +\infty),$$

$$Q(z) = \prod_{\nu=1}^{\infty} (1 - \beta_\nu z) \qquad (\beta_\nu > 0, \sum \beta_\nu < +\infty)$$

and such that both $P(z)/f(-z)$ and $Q(z)f(z)$ are integral functions. Moreover, when $z > 0$, the function $P(z)/f(-z)$ does not vanish because, by assertion (vii) the coefficients of its expansion are real, non-negative, and $a_0^{-1} = P(0)/f(0) \neq 0$. Hence $f(z)/P(-z)$ is regular for negative values of z and

$$\frac{f(z)}{P(-z)} Q(z) = \frac{f(z)Q(z)}{P(-z)} = v(z)$$

is obviously an integral function. Similarly, $1/v(z)$ is an integral function so that $v(z)$ has no zeros.

Hence

$$v(z) = e^{\phi(z)},$$

which proves the theorem of Aissen, Schoenberg, and Whitney.

4. Proof of Schoenberg's conjecture. As (1) is totally positive, the coefficients of the function

$$f^*(z) = a_0 + a_2 z + a_4 z^2 + \ldots + a_{2n} z^n + \ldots .$$

also form a totally positive sequence. Applying Proposition A to both $f(z)$ and $f^*(z)$, we can then assert the existence of

1. two integral functions $\phi(z)$ and $\phi^*(z)$,

2. four convergent series of non-negative terms, say

$$\Sigma\alpha_\nu, \quad \Sigma\beta_\nu, \quad \Sigma\alpha_\nu^*, \quad \Sigma\beta_\nu^*,$$

such that

$$f(z) = e^{\phi(z)} \frac{\prod (1 + \alpha_\nu z)}{\prod (1 - \beta_\nu z)}, \tag{23}$$

$$f^*(z) = e^{\phi^*(z)} \frac{\prod (1+a^*_\nu z)}{\prod (1-\beta^*_\nu z)}. \tag{24}$$

Now

$$\frac{f(z)+f(-z)}{2} = f^*(z^2);$$

using (23) and (24), this becomes

$$e^{\phi(z)} \frac{\prod (1+a_\nu z)}{\prod (1-\beta_\nu z)} + e^{\phi(-z)} \frac{\prod (1-a_\nu z)}{\prod (1+\beta_\nu z)} = 2e^{\phi^*(z^2)} \frac{\prod (1+a^*_\nu z^2)}{\prod (1-\beta^*_\nu z^2)}. \tag{25}$$

Put $\phi(z) = \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \dots$ and

$$u(z) = e^{2(\gamma_1 z + \gamma_3 z^3 + \gamma_5 z^5 + \dots)} \prod_{\nu=1}^{\infty} \frac{(1+a_\nu z)(1+\beta_\nu z)}{(1-a_\nu z)(1-\beta_\nu z)}; \tag{26}$$

equation (25) is then equivalent to

$$u(z) + 1 = 2e^{\phi^*(z^2)-\phi(-z)} \prod_{\nu=1}^{\infty} \frac{(1+a^*_\nu z^2)(1+\beta_\nu z)}{(1-\beta^*_\nu z^2)(1-a_\nu z)}. \tag{27}$$

This relation, together with (26), gives us complete information on the distribution of the values 0, ∞, and -1 of the meromorphic function $u(z)$.

The value -1 has an exponent of convergence not exceeding 2, whereas both the zeros and the poles have an exponent of convergence not exceeding 1.

By a well-known theorem of Nevanlinna [4, p. 72], the order of $u(z)$ does not exceed 2, so that

$$0 = \gamma_3 = \gamma_5 = \gamma_7 = \dots.$$

Hence $f(z)$ is of the form

$$f(z) = Ce^{\gamma_1 z + \gamma_2 z^2 + \gamma_4 z^4 + \dots} \frac{\prod (1+a_\nu z)}{\prod (1-\beta_\nu z)} \qquad (C>0). \tag{28}$$

Now if $\epsilon > 0$, both $e^{\epsilon z} f(z)$ and $e^{\epsilon z}/f(-z)$ have the property (P), so that, by assertion (vii), the power series expansions of

$$u_1(z) = Ce^{(\epsilon+\gamma_1)z + \gamma_2 z^2 + \gamma_4 z^4 + \dots} \prod_{\nu=1}^{\infty} (1+a_\nu z),$$

and

$$u_2(z) = C^{-1} e^{(\epsilon+\gamma_1)z - \gamma_2 z^2 - \gamma_4 z^4 - \dots} \prod_{\nu=1}^{\infty} (1+\beta_\nu z),$$

have all their coefficients non-negative.

Hence, putting

$$M_1(r) = \max_{|z|=r} |u_1(z)|, \qquad M_2(r) = \max_{|z|=r} |u_2(z)|,$$

we have

$$M_1(r) = u_1(r), \qquad M_2(r) = u_2(r),$$

and

$$M_1(r)M_2(r) = e^{2(\epsilon+\gamma_1)r}\prod_{\nu=1}^{\infty}(1+\alpha_\nu r)(1+\beta_\nu r). \tag{29}$$

As

$$M_2(r) \geqslant M_2(0) = C^{-1},$$

we see that the increase of $M_1(r)$ is dominated by the increase of an integral function of order 1. Hence

$$0 = \gamma_2 = \gamma_4 = \gamma_6 = \ldots.$$

We complete the proof of Schoenberg's conjecture by proving $\gamma = \gamma_1 \geqslant 0$. If γ were negative, we could choose $\epsilon > 0$ so that

$$\gamma + \epsilon < 0. \tag{30}$$

Now the increase of

$$\prod_{\nu=1}^{\infty}(1+\alpha_\nu r)(1+\beta_\nu r),$$

is at most of the minimal type of order 1. Hence (29) and (30) would imply

$$\lim_{r\to\infty} M_1(r)M_2(r) = 0,$$

which is contrary to the maximum-modulus theorem.

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Syracuse University

THE BOOLEAN ALGEBRA OF REGULAR OPEN SETS

R. S. PIERCE

1. Introduction. Let S be a completely regular topological space. Let $C(S)$ denote the set of bounded, real-valued, continuous functions on S. It is well known that $C(S)$ forms a distributive lattice under the ordinary pointwise joins and meets. For any distributive lattice L and any ideal $I \subseteq L$, a quasi-ordering of L can be defined as follows: $f \supseteq g$ if, for all $h \in L$, $f \cap h \in I$ implies $g \cap h \in I$. If equivalent elements under this quasi-ordering are identified, a homomorphic image of L is obtained. In this paper, the particular case where L is $C(S)$ and I is any principal ideal will be studied. It will be shown that the homomorphic image obtained in this way is isomorphic to a sub-lattice of $\mathfrak{B}(S)$, the Boolean algebra[1] of regular open sets of S. This homomorphic image will be denoted by $\mathfrak{L}(S)$; it does not depend on the choice of the principal ideal I. For the case where S is a normal space, a topological characterization of $\mathfrak{L}(S)$ will be obtained. Finally it will be proved that $\mathfrak{B}(S)$ is isomorphic to the normal completion (see [1]) of $\mathfrak{L}(S)$. Incidentally these results prove, without using transfinite methods, that for any completely regular space S, $\mathfrak{B}(S)$ is determined to within isomorphism by $C(S)$, a fact which could also be inferred from the theorem of Kaplansky [3] that a compact Hausdorff space S is determined to within homeomorphism by $C(S)$.

2. A congruence relation. In everything that follows, S will denote a completely regular topological space. Denote by $\leqslant$, $\cap$, $\cup$ the ordinary pointwise inequality, meet and join operations in $C(S)$.

Definition 2.1. $f \supseteq g$ if, for all $h \in C(S)$, $f \cap h \leqslant 0$ implies[2] $g \cap h \leqslant 0$. Write $f \sim g$ if $f \supseteq g$ and $g \supseteq f$.

LEMMA 2.1. *On $C(S)$, $\supseteq$ is a quasi-ordering and $\sim$ is a congruence relation.*

Proof. If $f \supseteq g$ and $g \supseteq h$, then $f \supseteq h$ is an immediate consequence of the definition. Also $f \supseteq f$ is clear. Thus $\supseteq$ is a quasi-ordering and consequently $\sim$ is an equivalence relation (Birkhoff [1]).

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[1] An open subset of a topological space is called regular open if it coincides with the interior of its closure. It is well known (see Birkhoff [1, p. 177]) that the regular open sets form a complete Boolean algebra.

[2] This definition puts the zero function in a distinguished position—a position which the lattice structure of $C(S)$ does not support. It is easy to see however that if 0 is replaced by an arbitrary (fixed) $f_0 \in C(S)$, all the results of the paper go through with only slight modification.

Let $f \supseteq g$ and f_1 be any continuous function; then $f \cap f_1 \supseteq g \cap f_1$ and $f \cup f_1 \supseteq g \cup f_1$. For suppose $(f \cap f_1) \cap h \leqslant 0$. Then $f \cap (f_1 \cap h) \leqslant 0$, so since $f \supseteq g$, $g \cap (f_1 \cap h) \leqslant 0$. Thus $(g \cap f_1) \cap h \leqslant 0$ and, since h was arbitrarily chosen, $f \cap f_1 \supseteq g \cap f_1$. Next suppose $(f \cup f_1) \cap h \leqslant 0$. This means $f_1 \cap h \leqslant 0$, $f \cap h \leqslant 0$ and, because $f \supseteq g$, $g \cap h \leqslant 0$. Hence $(g \cup f_1) \cap h = (g \cap h) \cup (f_1 \cap h) \leqslant 0$ and again by definition $f \cup f_1 \supseteq g \cup f_1$.

It follows that if $f \sim g$, then $f \cap f_1 \sim g \cap f_1$ and $f \cup f_1 \sim g \cup f_1$ for any $f_1 \in C(S)$. Thus $\sim$ is a congruence relation and the proof of the lemma is complete.

LEMMA 2.2. *$f \supseteq g$ if and only if*[3] $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}^-$.

Proof. First suppose $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}^-$. Let $h \in C(S)$ be such that $f \cap h \leqslant 0$. It will be shown that this implies $g \cap h \leqslant 0$, so (by definition) $f \supseteq g$.

If $x_0 \in \{x \mid g(x) > 0\}^-$, then by hypothesis, $x_0 \in \{x \mid f(x) > 0\}^-$. This means that if N is a neighbourhood of x_0, there is an $x \in N$ with $f(x) > 0$. Since $f \cap h \leqslant 0$, it follows that $h(x) \leqslant 0$. Thus $x_0 \in \{x \mid h(x) \leqslant 0\}^- = \{x \mid h(x) \leqslant 0\}$ and $h(x_0) \leqslant 0$. Consequently $(g \cap h)(x_0) \leqslant h(x_0) \leqslant 0$. If $x_0 \notin \{x \mid g(x) > 0\}^-$, then $g(x_0) \leqslant 0$ and $(g \cap h)(x_0) \leqslant g(x_0) \leqslant 0$. Hence for all points x_0 of S, $(g \cap h)(x_0) \leqslant 0$; that is, $g \cap h \leqslant 0$.

Now suppose $f \supseteq g$. Let $M = \{x \mid f(x) > 0\}^{-c} \cap \{x \mid g(x) > 0\}$. Clearly, M is open. Also if $y \in M$, then $f(y) \leqslant 0$ and $g(y) > 0$. To show that $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}^-$, it suffices to prove that M is empty, since then $\{x \mid f(x) > 0\}^- \supseteq \{x \mid g(x) > 0\}$, and taking closures on both sides gives the desired result.

Suppose M is not empty. Let $x_0 \in M$. By complete regularity, $h \in C(S)$ exists so that $h(x_0) = 1$ and $h(y) = 0$ for all $y \notin M$. Then $(g \cap h)(x_0) = \min\{1, g(x_0)\} > 0$, so $g \cap h$ non $\leqslant 0$. On the other hand, $(f \cap h)(y) \leqslant f(y) \leqslant 0$ if $y \in M$, while $(f \cap h)(y) \leqslant h(y) = 0$ if $y \notin M$. Thus $f \cap h \leqslant 0$. This contradicts $f \supseteq g$ and shows that M is empty.

Denote by $\mathfrak{L}(S)$ the set of equivalence classes under $\sim$. Then by Lemma 2.1, $\mathfrak{L}(S)$ is partially ordered under $\supseteq$ and, with this ordering, it is a distributive lattice. If $f \in C(S)$, let A_f denote the congruence class containing f.

LEMMA 2.3. *$\mathfrak{L}(S)$ is isomorphic to the sub-lattice of the Boolean algebra of regular open sets of S which consists of all sets of the form $I(\{x \mid f(x) > 0\}^-)$ where $f \in C(S)$.*

Proof. The mapping $A_f \to I(\{x \mid f(x) > 0\}^-)$ is one-to-one by Lemma 2.2. Also[4]

[3] For any set $N \subseteq S$, N^- will denote the closure of N, N^c the complement of N in S, while the interior of N, N^{-c}, will be represented by $I(N)$.

[4] The lattice operations in the Boolean algebra of regular open sets are given by $P \wedge Q = I(\overline{P \cap Q}^-) = P \cap Q$, $P \vee Q = I(\overline{P^- \cup Q^-})$ and $P' = P^{-c}$.

$$A_{f\cap g} \to I(\{x \mid (f \cap g)(x) > 0\}^-) = I([\{x \mid f(x) > 0\} \cap \{x \mid g(x) > 0\}]^-)$$
$$= I(\{x \mid f(x) > 0\}^-) \wedge I(\{x \mid g(x) > 0\}^-),$$
$$A_{f\cup g} \to I(\{x \mid (f \cup g)(x) > 0\}^-) = I(\{x \mid f(x) > 0\}^- \cup \{x \mid g(x) > 0\}^-)$$
$$= I(\{x \mid f(x) > 0\}^-) \vee I(\{x \mid g(x) > 0\}^-).$$

The following lemma gives information on the image of $\mathfrak{L}(S)$ which will be needed later.

LEMMA 2.4. *Let P be a regular open set of S.*
(a) *If $P \neq 0$ (the null set), $f \in C(S)$ exists so that $0 \neq I(\{x \mid f(x) > 0\}^-) \subseteq P$.*
(b) *If $P \neq S$ (the unit set), $f \in C(S)$ exists so that $S \neq I(\{x \mid f(x) > 0\}^-) \supseteq P$.*

Proof. (a) Let $x_0 \in P$. By complete regularity, choose $f \in C(S)$ so that $f(x_0) = 1$ and $f(y) = 0$ if $y \notin P$. Then $x_0 \in \{x \mid f(x) > 0\} \subseteq P$ so $x_0 \in I(\{x \mid f(x) > 0\}^-) \subseteq I(P^-) = P$.

(b) $P^- \neq S$ since otherwise $P = I(P^-) = I(S) = S$. Let $x_0 \in P^{-c}$. Choose $f \in C(S)$ so that $f(x_0) = -1$ and $f(y) = 1$ for all $y \in P^-$. Then

$$P \subseteq \{x \mid f(x) > 0\} \subseteq I(\{x \mid f(x) > 0\}^-) \subseteq \{x \mid f(x) \geqslant 0\} \neq S.$$

It may happen that the image of $\mathfrak{L}(S)$ under the mapping of Lemma 2.3 coincides with the whole Boolean algebra of regular open sets. For this to be the case, it is clearly necessary and sufficient that every regular open set P of S be of the form $P = I(\{x \mid f(x) > 0\}^-)$ for some $f \in C(S)$. This condition prevails in two cases of special importance. The first is that in which S is a metric space. For then, if P is a regular open set and $f \in C(S)$ is defined by $f(x) = \rho(x, P^c)$, ρ being the metric of S, $P = \{x \mid f(x) > 0\} = I(\{x \mid f(x) > 0\}^-)$ and the criterion is fulfilled. The second important case is that of an extremal space, that is, a space in which the closure of every open set is open. In such a space, the regular open sets are evidently just the open and closed sets. Hence the characteristic function ϕ_P of a regular open set P is continuous and $P = \{x \mid \phi_P(x) > 0\} = I(\{x \mid \phi_P(x) > 0\}^-)$. An important consequence is:

THEOREM 2.1. *The Boolean algebra of regular open sets of a completely regular topological space S is isomorphic to the complete Boolean algebra obtained by reducing the normal completion of $C(S)$ modulo the congruence of Definition* 2.1.

Proof. By the theorem of Dilworth [**2**], the normal completion of $C(S)$ is isomorphic to $C(\mathfrak{S})$ where $\mathfrak{S}$ is the Boolean space associated with $\mathfrak{B}(S)$, the Boolean algebra of regular open sets of S. Then (see Stone [**4**; **5**]) $\mathfrak{S}$ is extremal and its Boolean algebra of open and closed sets is isomorphic to $\mathfrak{B}(S)$. This means that S and $\mathfrak{S}$ have the same Boolean algebra of regular open sets. The theorem then follows from the above remark.

In spite of the above results, it is not true that, for all completely regular spaces S, the image of $\mathfrak{L}(S)$ is the whole Boolean algebra of regular open sets. This is shown by the following example.

Let S be the set W, consisting of all ordinal numbers $\leqslant \Omega$ (the least ordinal of the third class), together with the real interval $I = [0, 1]$, where Ω is identified with 0. Simply order S by defining $\alpha < x$ for all $\alpha \in W$ and $x \in I$, and by retaining the usual order in W and I. Impose the interval topology (see [1]) on S. It is easily seen that S is completely regular. Now the interval $P = \{a \in S \mid a < \Omega\}$ is a regular open set of S. Suppose $f \in C(S)$ exists with $P = I(\{a \mid f(a) > 0\}^-)$. Then $f(a) \leqslant 0$ holds for all $a \geqslant 0 = \Omega$. Let $\alpha_n = \sup \{a \mid f(a) \geqslant 1/n\}$. For all n, $\alpha_n < \Omega$ since $\{a \in S \mid f(a) < 1/n\}$ is open and contains I. Thus α_n is an ordinal of the second class. $\alpha = \sup \alpha_n$, as the limit of a sequence of ordinals of the second class, is itself an ordinal of the second class. Hence $\alpha < \Omega$. Because

$$\{a \mid f(a) > 0\} = \bigcup_n \{a \mid f(a) \geqslant 1/n\} \subseteq \bigcup_n \{a \mid a \leqslant \alpha_n\} \subseteq \{a \mid a \leqslant \alpha\},$$

it follows that $I(\{a \mid f(a) > 0\}^-) \subseteq \{a \mid a < \alpha\} \neq P$. This is a contradiction, showing that no such $f \in C(S)$ can exist.

The preceding example indicates the need for a topological characterization of $\mathfrak{L}(S)$. For the important case where S is a normal space, the following theorem answers this need.

THEOREM 2.2. *Let S be a normal topological space. In the mapping $M \to I(M^-)$ of the lattice of open sets onto the Boolean algebra $\mathfrak{B}(S)$, $\mathfrak{L}(S)$ is the image of the sub-lattice of all open sets which are the countable unions of closed sets (that is, open F_σ sets).*

Proof. $\mathfrak{L}(S)$ consists of all sets of the form $P = I(\{x \mid f(x) > 0\}^-)$ where $f \in C(S)$. But

$$\{x \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \mid f(x) \geqslant 1/n\}$$

is an open F_σ set. Thus every $P \in \mathfrak{L}(S)$ is the image of an open F_σ set.

On the other hand, suppose $P = I(M^-)$ where M is open and

$$M = \bigcup_{n=1}^{\infty} F_n$$

is the countable union of the closed sets F_n. By Urysohn's lemma, for each n, it is possible to select $f_n \in C(S)$ so that $f_n(x) = 1$ for all $x \in F_n$, $f_n(x) = 0$ for all $x \in M^c$ and $0 \leqslant f_n \leqslant 1$. Define

$$f = \sum_{n=1}^{\infty} (2^{-n}) f_n.$$

Then $f \in C(S)$ and $f(x) = 0$ if $x \in M^c$. If $x \in M$, then $x \in F_n$ for some n, whence

$$f(x) \geqslant 2^{-n} f_n(x) = 2^{-n} > 0.$$

Consequently $M = \{x \mid f(x) > 0\}$ and $P = I(M^-) = I(\{x \mid f(x) > 0\}^-) \in \mathfrak{L}(S)$ Thus the image of any open F_σ is in $\mathfrak{L}(S)$ and the proof is complete.

3. Completion. Having constructed the homomorphic image of $C(S)$, it will now be shown that the normal completion of $\mathfrak{L}(S)$ is isomorphic to the Boolean algebra of regular open sets.

Let it be recalled that the normal completion of a lattice $\mathfrak{L}$ consists of the normal subsets (or closed ideals) of $\mathfrak{L}$ ordered by inclusion, that is, those subsets which contain all lower bounds to the set of their upper bounds. The mapping $A \to \{B \in \mathfrak{L} \mid B \subseteq A\}$ imbeds $\mathfrak{L}$ as a sub-lattice of its normal completion (for proofs, see Birkhoff [1]).

LEMMA 3.1.[5] *Let $\mathfrak{L}$ be a sub-lattice of the complete Boolean algebra $\mathfrak{B}$. Suppose, moreover, that the following dual conditions are satisfied: If $X \in \mathfrak{B}$ and $X \neq Z$ (zero of $\mathfrak{B}$), then $A \in \mathfrak{L}$ exists with $Z \neq A \subseteq X$. If $X \in \mathfrak{B}$ and $X \neq I$ (unit of $\mathfrak{B}$), then $A \in \mathfrak{L}$ exists with $I \neq A \supseteq X$. Then the normal completion of $\mathfrak{L}$ is isomorphic to $\mathfrak{B}$.*

Proof. For $X \in \mathfrak{B}$, let $\mathfrak{N}_X = \{A \in \mathfrak{L} \mid A \subseteq X\}$. The first task is to show that $\mathfrak{N}_X$ is a normal subset of $\mathfrak{L}$.

First note that, if $A \supseteq B$ holds for some $A \in \mathfrak{L}$ and all $B \in \mathfrak{N}_X$, then $A \supseteq X$. For otherwise $A' \cap X \neq Z$ and, by hypothesis, $B \in \mathfrak{L}$ exists with $A' \cap X \supseteq B \neq Z$. This means A non $\supseteq B$ while $B \subseteq X$ which contradicts the assumption on A. Thus the upper bounds in $\mathfrak{L}$ of all $B \in \mathfrak{N}_X$ are precisely those A satisfying $A \supseteq X$. As a result, the preceding argument can be dualized to show that a lower bound, A, of the set of all upper bounds of $\mathfrak{N}_X$ satisfies $A \subseteq X$. Consequently $\mathfrak{N}_X$ is normal.

Next, every normal subset of $\mathfrak{L}$ is of the form $\mathfrak{N}_X$ for some $X \in \mathfrak{B}$. In fact, if $\mathfrak{M}$ is normal, let $X = \bigcup \mathfrak{M}$ (sup in $\mathfrak{B}$). Then if $A \in \mathfrak{M}$, $A \subseteq X$, so $\mathfrak{M} \subseteq \mathfrak{N}_X$. On the other hand, by the definition of X, if $C \supseteq B$ holds for all $B \in \mathfrak{M}$, $C \supseteq X$. Thus $A \in \mathfrak{N}_X$ (that is, $A \subseteq X$) implies $A \subseteq C$ for all upper bounds of $\mathfrak{M}$. Since $\mathfrak{M}$ is normal, this means $A \in \mathfrak{M}$, so $\mathfrak{N}_X \in \mathfrak{M}$.

The above argument shows $X \to \mathfrak{N}_X$ is a mapping from $\mathfrak{B}$ onto the normal completion of $\mathfrak{L}$. The mapping obviously preserves order: $X_1 \subseteq X_2$ implies $\mathfrak{N}_{X_1} \subseteq \mathfrak{N}_{X_2}$. To complete the proof, it is sufficient to show that this mapping is one-to-one.

Suppose X_1 non $\subseteq X_2$. Then $Z \neq X_1 \cap X_2'$, so $A \in \mathfrak{L}$ exists with $Z \neq A \subseteq X_1 \cap X_2'$. It follows that $A \in \mathfrak{N}_{X_1}$ and $A \notin \mathfrak{N}_{X_2}$, that is, $\mathfrak{N}_{X_1}$ non $\subseteq \mathfrak{N}_{X_2}$.

COROLLARY. *If S_1 and S_2 are two completely regular topological spaces such that $\mathfrak{L}(S_1)$ is isomorphic to $\mathfrak{L}(S_2)$, then S_1 and S_2 have isomorphic Boolean algebras of regular open sets.*

According to Lemma 2.4 the above principle is applicable to the image of $\mathfrak{L}(S)$ in the complete Boolean algebra of regular open sets. The result of this application is:

[5] This lemma, in essence, was presented by Professor Dilworth during his 1950–51 seminar at C.I.T.

THEOREM 3.1. *The Boolean algebra of regular open sets of a completely regular topological space S is isomorphic to the normal completion of the lattice obtained by reducing C(S) modulo the congruence of Definition* 2.1.

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California Institute of Technology

CONTINUOUS SOLUTIONS OF THE FUNCTIONAL EQUATION $f^n(x) = f(x)$

G. M. EWING AND W. R. UTZ

In this note the authors find all continuous real functions defined on the real axis and such that for an integer $n > 2$, and for each x,

$$f^n(x) = f(x). \tag{1}$$

The symbol f^n denotes f iterated n times.

The following two classes of functions occur as solutions.

CLASS I.

(a_1) *The function $f(x)$ is continuous for all real x,*

(b_1) *$f(x) \equiv x$ on a connected subset S of the x-axis, and*

(c_1) *$g \leqslant f(x) \leqslant G$, in which g and G denote, respectively, the infinum and supremum of $f(x)$ on S.*

The set S must be a point, a closed interval, a closed ray, or the entire x-axis. Thus Class I includes all constants and the function x. If S is a closed interval $[a, b]$ then $f(x)$ is arbitrary outside of $[a, b]$ except for continuity and the condition $a \leqslant f(x) \leqslant b$. If S is a ray, $f(x)$ is similarly described.

CLASS II.

(a_2) *The function $f(x)$ is continuous for all real x and either,*

(b_2) *$f^2(x) \equiv x$, or*

(c_2) *$f^2(x) = x$ on a non-degenerate closed interval $[a, b]$, $f(a) = b$, $f(b) = a$, and $a \leqslant f(x) \leqslant b$.*

A function satisfies (b_2) if and only if $y = f(x)$ implies $x = f(y)$. Its graph is, accordingly, symmetric with respect to the line $y = x$. If $f(x)$ is a solution of (b_2), then the inverse of the transformation $x \to f(x)$ is clearly single valued and continuous. Hence the transformation $x \to f(x)$ defines a homeomorphism of the x-axis onto itself.

One can easily see that Classes I and II have only the function x in common.

LEMMA 1. *$f(x)$ is a continuous solution of $f^2(x) = f(x)$ if, and only if, $f(x)$ is of Class* I.

Proof. That every Class I function is a solution is easily verified. Conversely, if $f(x)$ is a continuous solution then $x = f(r)$ satisfies $f(x) = x$ for every real r. In case $f(x) = x$ has only one solution, $f(x)$ is constant and hence of Class I. If $f(x) = x$ has two solutions a and b, $a < b$, then $f(a) = a$ and $f(b) = b$; and given c between a and b there exists, from the continuity of $f(x)$,

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a number m, $a < m < b$, such that $f(m) = c$. It follows that $f(c) = ff(m) = f(m) = c$, and hence if $f(x) = x$ at the ends of an interval the relation holds identically on the interval. The maximal set S on which $f(x) = x$ is thus connected. The continuity of $f(x)$ implies that S is closed. Finally $f(x)$ has property (c_1), for if there exists an r not in S such that $f(r)$ does not satisfy (c_1), the relation $f(x) = ff(x) = f^2(x)$ with $x = r$, contradicts the fact that S is maximal.

LEMMA 2. *If $f^n(x) = x$ on a non-degenerate closed connected subset S of the real axis and if $f(x)$ maps S continuously into S, then*

(i) *$f(x)$ is a homeomorphism of S onto itself,*

(ii) *if S is an interval $[a, b]$, $f(x) \equiv x$ on S or $f^2(x) \equiv x$ on S and $f(x)$ is equivalent to a reflection of $[a, b]$ about the single fixed point p,*

(iii) *if S is a ray, $f(x) \equiv x$ on S, and*

(iv) *if S is the entire axis, $f(x) \equiv x$ on S or $f^2(x) \equiv x$ on S and $f(x)$ is equivalent to a reflection of S about the single fixed point p.*

Proof. Conclusions (i) for the case of an interval and (ii) are special cases of results in Whyburn [**2**, pp. 240, 264].

If S is a ray, the mapping [**2**, p. 240] is (1–1) and onto. Thus $f(x)$ is monotone on the ray and the end point is fixed under $f(x)$. If there were an interior point b of the ray such that $f(b) \neq b$, the monotonicity of $f(x)$ would imply that $f^n(b) \neq b$. Hence (iii), which implies (i) for the ray.

If S is the real axis, the mapping is again (1–1) and onto and $f(x)$ is monotone. If $f(x)$ increases with x, we see that $f(x) \equiv x$ by the argument employed for the ray. If $f(x)$ is monotone decreasing its graph cuts $y = x$ in exactly one point and $f(x)$ is topologically equivalent to a reflection of the x-axis about the abscissa of this point.

COROLLARY 1. *If n is odd, the functional equation $f^n(x) = x$ has only the function x as a continuous solution. If n is even, the continuous solutions of $f^n(x) = x$ are those of $f^2(x) = x$.*

Proof. By conclusion (iv) of Lemma 2, there are two possibilities. If $n = 2m + 1$ and if $f^2(x) \equiv x$ then $f^{2m}(x) \equiv f^2 f^2 \ldots f^2(x) \equiv x$, and hence $f^{2m+1}(x) \equiv f(x) \equiv x$. If n is even, the stated result is immediate from Lemma 2 since $f(x) \equiv x$ is a solution of $f^2(x) = x$.

THEOREM 1. *The continuous real solutions of $f^n(x) = f(x)$, $n \geqslant 2$, are the functions of Class* I *if n is even and the functions of Classes* I *and* II *if n is odd.*

Proof. If $f(x)$ is of Class I then $f^2(x) = f(x)$. Whence

$$f^3(x) = f^2(x) = f(x), \ldots .$$

If $f(x)$ is of Class II we verify that $f^3(x) = f(x)$. Then

$$f^3(x) = f^3(x) = f(x), \ f^7(x) = f^5(x) = f(x), \ldots .$$

Conversely, let $f(x)$ be a continuous solution of $f^n(x) = f(x)$. Then

$$f^{n-1} f^{n-1}(x) = f^{n-2} f^n(x) = f^{n-2} f(x) = f^{n-1}(x)$$

so that $f^{n-1}(x)$ if of Class I by Lemma 1. Let S be the maximal subset of the x-axis on which $f^{n-1}(x) \equiv x$.

If S is a point, then $f^{n-1}(x) \equiv c$ and $f(x) \equiv ff^{n-1}(x) \equiv f(c)$, so that $f(x)$ is of Class I.

If S is the closed interval $[a, b]$ then $f^{n-1}(x) \equiv x$ on $[a, b]$ and $a \leqslant f^{n-1}(x) \leqslant b$ by (b_1) and (c_1). Moreover $a \leqslant f(x) \leqslant b$, as a consequence of the relations

$$f(x) = f^n(x) = f^{n-1}f(x). \tag{2}$$

Thus $f(x)$ maps the real axis into $[a, b]$. In particular $[a, b]$ goes into $[a, b]$ and Lemma 2 is applicable. If $f(x) \equiv x$ on $[a, b]$, $f(x)$ is of Class I. The other possibility is that $f^2(x) \equiv x$ on $[a, b]$, and that $f(a) = b$, $f(b) = a$, in which event $f(x)$ is of Class II.

If $f^{n-1}(x) \equiv x$ on a ray, we see that $f(x)$ maps the reals into $[a, \infty]$ or $[-\infty, b]$. Hence $f(x) \equiv x$ on the ray by Lemma 2 and is of Class I.

Finally, if S is the x-axis, the desired conclusion is given by Corollary 1.

The functional equation

$$f^n(x) = f^m(x), \tag{3}$$

m and n integers, $1 < m < n$, has among its continuous solutions the functions of Classes I and II if $m + n$ is even and those of Class I if $m + n$ is odd. In either case, if $f(x)$ is a solution of (3) there exists an integer k such that $f^k(x)$ is of Class I. However, each equation (3) has continuous solutions in neither of our classes. For example $f^3(x) = f^2(x)$ has the solution

$$f(x) = \begin{cases} \sin \pi x & x > 1 \\ 0 & |x| \leqslant 1 \\ -x-1 & -2 \leqslant x < -1 \\ 1 & x < -2. \end{cases}$$

Let $y = f(x)$ denote a transformation of a topological space X into itself. A necessary and sufficient condition that $f^2(x) = f(x)$ have $f(x)$ as a continuous solution is that the set S of fixed points under $f(x)$ be non-vacuous and that $f(x)$ map X continuously into S. That $f(x)$ be a continuous solution of $f^2(x) = f(x)$ is a restriction on both S and $f(x)$. If X is the real axis these restrictions are given by our Lemma 1. In general the solutions of $f^2(x) = f(x)$ are the "retractions" of the space X. These have been studied by Borsuk [1] but results for higher-dimensional or abstract cases comparable to those of Lemma 1 do not seem to be available and appear difficult to achieve.

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The Sandia Corporation
and
The University of Missouri

GENERATING FUNCTIONS FOR BESSEL AND RELATED POLYNOMIALS

E. D. RAINVILLE

1. Introduction. Krall and Frink [4] aroused interest in what they term Bessel polynomials. They studied in some detail what may, in hypergeometric form, be written as

$$y_n(x, a, b) = {}_2F_0(-n, a-1+n; -; -x/b). \tag{1}$$

The simple Bessel polynomial is the special case $a = b = 2$. In essence the polynomials considered by Krall and Frink, and by others [1; 3], are the terminating ${}_2F_0$'s with numerator parameters $(-n)$ and $(c+n)$, n a non-negative integer. We shall therefore work with

$$ {}_2F_0(-n, c+n; -; -x) \tag{2}$$

for brevity. From (2), the polynomial (1) is easily obtained.

Burchnall [1] produced a convergent generating function for $y_n(x, a, b)$. It is the purpose of this note to obtain relations which yield in simple special instances two generating functions, one of them Burchnall's, for Bessel polynomials.

2. Generating functions of Burchnall type. Consider the polynomials

$$\psi_n(x) = {}_{q+2}F_p\left[\begin{matrix} -n, c+n, 1-\beta_1-n, 1-\beta_2-n, \ldots, 1-\beta_q-n; \\ 1-\alpha_1-n, 1-\alpha_2-n, \ldots, 1-\alpha_p-n; \end{matrix} (-1)^{p+q+1}x\right] \tag{3}$$

in which the c and all the α's and β's are independent of n, and in which n is a non-negative integer. The α's and β's are not to be non-positive integers, but are otherwise unrestricted.

If in (3) no α's and no β's are used, the resulting polynomial is that of (2), the generalized Bessel polynomial.

We now prove that

$$(1-4xt)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{(1-4xt)}}\right]^{c-1} {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \frac{2t}{1+\sqrt{(1-4xt)}}\right] = \sum_{n=0}^{\infty} \frac{\psi_n(x)(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n t^n}{n!\,(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n}, \tag{4}$$

in which $(1-4xt)^{\frac{1}{2}} \to 1$ as $t \to 0$. Burchnall's [1] generating function is a

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special case of (4) in which there are no α's and no β's, x is to be replaced by (x/b) and t by $(\frac{1}{2}bt)$. In Burchnall's paper the misprint, $\{\frac{1}{2} - \frac{1}{2}(1 - 2xt)^{\frac{1}{2}}\}^{2-a}$ where $\{\frac{1}{2} + \frac{1}{2}(1 - 2xt)^{\frac{1}{2}}\}^{2-a}$ was intended, should be corrected.

For our proof of (4) we need a simple result from the theory of the ${}_2F_1$.

LEMMA.

$$ {}_2F_1(\gamma, \gamma + \tfrac{1}{2}; 2\gamma; 4z) = (1 - 4z)^{-\frac{1}{2}}\left[\frac{2}{1 + \sqrt{(1 - 4z)}}\right]^{2\gamma-1}. $$

To obtain the result in the lemma, use Gauss's formula

$$ {}_2F_1(a, b; 2b; 4x(1 + x)^{-2}) = (1 + x)^{2a}\, {}_2F_1(a, a + \tfrac{1}{2} - b; b + \tfrac{1}{2}; x^2), $$

for which see Magnus and Oberhettinger [5], put $b = \gamma$, $s = \gamma + \frac{1}{2}$, $x(1 + x)^{-2} = z$, and use the fact that a ${}_1F_0$ is a binomial.

Since $(1 - \alpha - n)_k = (-1)^k(\alpha)_n/(\alpha)_{n-k}$, it follows that

$$ \frac{\psi_n(x)(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{n!\,(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} = \sum_{k=0}^{n} \frac{(c)_{n+k}(\alpha_1)_{n-k}(\alpha_2)_{n-k} \cdots (\alpha_p)_{n-k}\, x^k}{(c)_n(\beta_1)_{n-k}(\beta_2)_{n-k} \cdots (\beta_q)_{n-k}(n-k)!\,k!}. $$

Therefore,

$$ \begin{aligned} &\sum_{n=0}^{\infty} \frac{\psi_n(x)(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n t^n}{n!\,(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c)_{n+k}(\alpha_1)_{n-k}(\alpha_2)_{n-k} \cdots (\alpha_p)_{n-k}\, x^k t^n}{(c)_n(\beta_1)_{n-k}(\beta_2)_{n-k} \cdots (\beta_q)_{n-k}(n-k)!\,k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c)_{n+2k}(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n\, x^k t^{n+k}}{(c)_{n+k}(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n\, n!\,k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c+n)_{2k}(xt)^k}{(c+n)_k\,k!} \cdot \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n t^n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n\, n!} \\ &= \sum_{n=0}^{\infty} {}_2F_1(\tfrac{1}{2}c + \tfrac{1}{2}n, \tfrac{1}{2}c + \tfrac{1}{2}n + \tfrac{1}{2}; c + n; 4xt) \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n\, t^n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n\, n!} \\ &= \sum_{n=0}^{\infty} (1 - 4xt)^{-\frac{1}{2}}\left[\frac{2}{1 + \sqrt{(1 - 4xt)}}\right]^{c+n-1} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n\, t^n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n\, n!} \\ &= (1 - 4xt)^{-\frac{1}{2}}\left[\frac{2}{1 + \sqrt{(1 - 4xt)}}\right]^{c-1} {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix} \frac{2t}{1 + \sqrt{(1 - 4xt)}}\right], \end{aligned} $$

which completes the derivation of equation (4). The method used is nothing like that in Burchnall's work.

If $p \leqslant q + 1$, as is true for the generalized Bessel polynomials, the generating function series, the ${}_pF_q$, even has a region of convergence near $t = 0$. For generating function purposes the luxury of convergence is, fortunately, not at all necessary.

3. Another generating function for Bessel polynomials. Consider next the polynomials

$$(5) \qquad \sigma_n(x) = {}_{p+2}F_q\left[\begin{matrix} -n, c+n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} -x\right],$$

of which the Bessel polynomials are again a special instance, namely that in which no α's and no β's are used. This extension of the generalized Bessel polynomials is, of course, distinct from that in equation (3).

We shall obtain the generating function relation

$$(6) \quad (1-t)^{-c}\,{}_{p+2}F_q\left[\begin{matrix} \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \frac{4xt}{(1-t)^2}\right] = \sum_{n=0}^{\infty} \frac{\sigma_n(x)(c)_n t^n}{n!},$$

which contains a divergent generating function, a ${}_2F_0$, for the Bessel polynomials, namely the choice of no α's and no β's. The series in (6) converge near $t = 0$ if $p \leqslant q - 1$.

We find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\sigma_n(x)(c)_n t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k (c+n)_k (c)_n (\alpha_1)_k (\alpha_2)_k \ldots (\alpha_p)_k (-x)^k t^n}{(\beta_1)_k (\beta_2)_k \ldots (\beta_q)_k\, k!\, n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(c)_{n+k} (\alpha_1)_k (\alpha_2)_k \ldots (\alpha_p)_k\, x^k t^n}{(\beta_1)_k (\beta_2)_k \ldots (\beta_q)_k\, k!\, (n-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c)_{n+2k} (\alpha_1)_k (\alpha_2)_k \ldots (\alpha_p)_k\, x^k t^{n+k}}{(\beta_1)_k (\beta_2)_k \ldots (\beta_q)_k\, k!\, n!} \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c+2k)_n\, t^n (c)_{2k} (\alpha_1)_k (\alpha_2)_k \ldots (\alpha_p)_k (xt)^k}{n!\, (\beta_1)_k (\beta_2)_k \ldots (\beta_q)_k\, k!} \\
&= \sum_{k=0}^{\infty} \frac{(c)_{2k} (\alpha_1)_k (\alpha_2)_k \ldots (\alpha_p)_k\, (xt)^k}{(\beta_1)_k (\beta_2)_k \ldots (\beta_q)_k\, k!\, (1-t)^{c+2k}} \\
&= (1-t)^{-c}\,{}_{p+2}F_q\left[\begin{matrix} \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}, \alpha_1, \alpha_2, \ldots \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \frac{4xt}{(1-t)^2}\right],
\end{aligned}$$

which completes the derivation of equation (6).

For $c = 1$, the equivalent of equation (6) has already appeared in Sister Celine's work [2].

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The University of Michigan

ON TWO-DIMENSIONAL BERNSTEIN POLYNOMIALS

P. L. BUTZER

1. Introduction. Let the function of two real variables $f(x, y)$ be given over the square

$$S: 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1.$$

Then the Bernstein polynomial in two variables x and y, corresponding to the function $f(x, y)$ is defined as

$$B^f_{n_1,n_2}(x, y) = \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} f\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) p_{\nu_1,n_1}(x)\ p_{\nu_2,n_2}(y). \tag{1}$$

where

$$p_{\nu,n}(u) = \binom{n}{\nu} u^\nu (1-u)^{n-\nu}.$$

Obviously

$$\sum_{\nu=0}^{n} p_{\nu,n}(u) = 1.$$

The main purpose of this paper will be to prove the following result.

THEOREM. *If the function $f(x, y)$ is bounded in the closed square S, then*

$$\lim_{n_1,n_2\to\infty} \frac{\partial^p}{\partial x_0^q\, \partial y_0^{p-q}} B^f_{n_1,n_2}(x_0, y_0) = \frac{\partial^p}{\partial x_0^q\, \partial y_0^{p-q}} f(x_0, y_0) \tag{2}$$

at every point (x_0, y_0) belonging to the open square $0 < x < 1$, $0 < y < 1$ where the pth total differential of $f(x, y)$ exists, provided n_1, n_2 tend to infinity in such a way that

$$0 < r \leqslant \frac{n_1 + 1}{n_2 + 1} \leqslant s < +\infty. \tag{3}$$

If, moreover, the partial derivatives of $f(x, y)$ of the first p orders exist and are continuous in S, then the relation (2) holds uniformly in S as n_1, n_2 approach infinity in any manner whatsoever. This latter result, which is used in establishing our theorem, was recently proven by Kingsley [1], but we shall deduce it in a more direct manner.

2. Preliminary results. If $\delta \geqslant n_1^{-\alpha}$, $0 < \alpha < \frac{1}{2}$ then for every $k > 0$ there is a constant C (depending only on α and k) such that

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$$\sum_{\left|\frac{\nu}{n}-u\right|\geqslant\delta} p_{\nu,n}(u) < Cn^{-2}, \qquad 0 \leqslant u \leqslant 1. \tag{4}$$

Furthermore, for a given $\gamma > 0$ and any $0 \leqslant u \leqslant 1$,

$$\sum_{\nu=0}^{n} |\nu - nu|^{\gamma}\, p_{\nu,n}(u) < C_1 n^{\frac{1}{2}\gamma} \tag{5}$$

where C_1 is a constant depending only on γ.

For the proofs of these inequalities see [2].

We now recall several facts on differentials. By definition, the pth total differential of $f(x, y)$ exists [3] at the point (x_0, y_0) if

$$f(x, y) = P(x, y) + g(x, y) \tag{6}$$

where $P(x, y)$ represents a polynomial of degree p, and

$$g(x, y) = \sum_{i=0}^{p} \alpha_i(x, y)(x - x_0)^{p-i}(y - y_0)^i = \sum_{i=0}^{p} \beta_i \tag{7}$$

for the sake of brevity, where the $\alpha_i(x, y)$ have zero as a unique double limit as $x \to x_0$, $y \to y_0$.

We now define

$$\underset{(x,h)}{\Delta^0} f(x, y) = f(x, y), \quad \underset{(x,h)}{\Delta^1} f(x, y) = f(x + h, y) - f(x, y)$$

and in general

$$\underset{(x,h)}{\Delta^p} f(x, y) = \sum_{i=1}^{p} (-1)^{p-i}\binom{p}{i} f(x + ih, y), \tag{8}$$

and similarly

$$\underset{(y,k)}{\Delta^q} f(x, y) = \sum_{j=1}^{q} (-1)^{q-j}\binom{q}{j} f(x, y + jk). \tag{9}$$

From (8) and (9) it follows that

$$\begin{aligned} \underset{(x,h)}{\Delta^p}\ \underset{(y,k)}{\Delta^q} f(x, y) &= \underset{(y,k)}{\Delta^q}\ \underset{(x,h)}{\Delta^p} f(x, y) \\ &= \sum_{i=0}^{p}\sum_{j=0}^{q} (-1)^{p+q-i-j}\binom{p}{i}\binom{q}{j} f(x + ih, y + jk). \end{aligned} \tag{10}$$

If the partial derivatives of $f(x, y)$ of order $\leqslant p + q$ all exist at (x_0, y_0) and are continuous in S, it can be shown that

$$\underset{(x_0,h)}{\Delta^p}\ \underset{(y_0,k)}{\Delta^q} f(x_0, y_0) = h^p k^q \frac{\partial^{p+q}}{\partial x_0^q\, \partial y_0^{p-q}} f(x_0 + p\theta_1 h, y + q\theta_2 k) \tag{11}$$

where $\theta_i = \theta_i(x_0, y_0; h, k)$ $(0 < \theta_i < 1\ ;\ i = 1, 2)$.

Applying (10), by induction we can show for $q = 0, 1, 2, \ldots, p$,

$$\begin{aligned} &\frac{\partial^p}{\partial x^q\, \partial y^{p-q}} B^f_{n_1,n_2}(x, y) \\ &= n_1(n_1 - 1)\ldots(n_1 - q + 1)n_2(n_2 - 1)\ldots(n_2 - p + q + 1) \\ &\quad \cdot \sum_{\nu_1=0}^{n_1-q}\sum_{\nu_2=0}^{n_2-p+q} \underset{(x,n_1^{-1})}{\Delta^q}\ \underset{(y,n_2^{-1})}{\Delta^{p-q}} f\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) p_{\nu_1,n_1-q}(x)\, p_{\nu_2,n_2-p+q}(y). \end{aligned} \tag{12}$$

3. The main theorem. We state Bernstein's Theorem for functions of two variables.

LEMMA 1. *If $f(x, y)$ is bounded in the square S, then at every point of continuity (x, y),*

$$\lim_{n_1,\, n_2 \to \infty} B^f_{n_1,\, n_2}(x, y) = f(x, y);$$

the result holding uniformly in x and y if $f(x, y)$ is continuous in S.

The proof follows simply by extending the proof of the one-dimensional Bernstein's Theorem to two dimensions.

LEMMA 2. *If all the partial derivatives of $f(x, y)$ of order $\leqslant p$ exist and are continuous in S, then*

$$\frac{\partial^p}{\partial x^q \partial y^{p-q}} B^f_{n_1,\, n_2}(x, y) \to \frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x, y) \tag{13}$$

uniformly in S as n_1, n_2 approach infinity in any manner whatever.

Proof. Applying the relations (12) and (11) we obtain

$$\begin{aligned} &\frac{\partial^p}{\partial x^q \partial y^{p-q}} B^f_{n_1,\, n_2}(x, y) \\ &= n_1(n_1 - 1) \ldots (n_1 - q + 1) n_2(n_2 - 1) \ldots (n_2 - p + q + 1) \\ &\quad \cdot \left(\frac{1}{n_1}\right)^q \left(\frac{1}{n_2}\right)^{p-q} \sum_{\nu_1=0}^{n_1-q} \sum_{\nu_2=0}^{n_2-p+q} \frac{\partial^p}{\partial x^q \partial y^{p-q}} f\left(\frac{\nu_1}{n_1} + q\,\frac{\theta_{\nu_1,\nu_2}}{n_1}, \frac{\nu_2}{n_2} + (p - q)\,\frac{\phi_{\nu_1,\nu_2}}{n_2}\right) \\ &\qquad \cdot p_{\nu_1, n_1-q}(x)\, p_{\nu_2, n_2-p+q}(y), \end{aligned} \tag{14}$$

where

$$0 < \theta_{\nu_1,\nu_2} < 1, \quad 0 < \phi_{\nu_1,\nu_2} < 1.$$

Now, in the relation (14), the product of the factors outside the double sum will approach 1 as n_1, n_2 approach infinity. We denote $\partial^p f / \partial x^q \partial y^{p-q}$ by $h(x, y)$ and observe that the difference between the double sum in (14) and

$$B^h_{n_1-q, n_2-p+q}(x, y)$$

approaches zero uniformly in x and y as $n_1, n_2 \to \infty$. But by the previous lemma, the last expression approaches

$$\frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x, y)$$

uniformly in x and y. Hence the limit relation (13) follows, proving the lemma.

LEMMA 3.

$$\frac{d^l}{du^l}\left[u^r(1 - u)^{n-r}\right] = Q(u) u^{r-l}(1 - u)^{n-r-l}$$

where

$$Q(u) = \sum_{i,j} n^i(\nu - nu)^j h^l_{i,j}(u), \qquad i, j \geq 0,\ 2i + j \leq l$$

and the $h^l_{i,j}(u)$ are polynomials in u independent of ν and n.

The proof of this lemma follows by induction, see [2].

We are now in a position to consider the theorem stated in the introduction.

Proof of Theorem. Since the pth total differential of $f(x, y)$ exists at (x_0, y_0), we have the representaion (6) with $P(x, y)$ a polynomial of degree p and $g(x, y)$ as defined by (7). We therefore obtain

$$\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^f_{n_1,n_2}(x_0, y_0) = \frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^P_{n_1,n_2}(x_0, y_0) + \frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^g_{n_1,n_2}(x_0, y_0)$$

and by Lemma 2 we only need to show that the second term on the right-hand side approaches zero. Using the definition of $g(x, y)$ in (7), it is sufficient to show that

$$\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^{\delta_k}_{n_1,n_2}(x_0, y_0) \to 0, \qquad k = 0, 1, 2, \ldots, p. \tag{15}$$

By Lemma 3, we find that

$$\begin{aligned}
&\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^{\delta_k}_{n_1,n_2}(x_0, y_0) \\
&= \sum_{2i_1+j_1 \leq q} h^q_{i_1,j_1}(x_0) \sum_{2i_2+j_2 \leq p-q} h^{p-q}_{i_2,j_2}(y_0) \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right)\left(\frac{\nu_1}{n_1} - x_0\right)^{p-k} \\
&\quad \cdot \left(\frac{\nu_2}{n_2} - y_0\right)^k n_1^{i_1}(\nu_1 - n_1x_0)^{j_1} n_2^{i_2}(\nu_2 - n_2y_0)^{j_2} \binom{n_1}{\nu_1} x_0^{\nu_1-q}(1 - x_0)^{n_1-\nu_1-q} \binom{n_2}{\nu_2} \\
&\quad \cdot y_0^{\nu_2-p+q}(1 - y_0)^{n_2-\nu_2-p+q}
\end{aligned}$$

and we can rewrite this in the form

$$\begin{aligned}
\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^{\delta_k}_{n_1,n_2}(x_0, y_0) &= \sum_{2i_1+j_1 \leq q} h^*_{i_1,j_1}(x_0) \sum_{2i_2+j_2 \leq p-q} h^*_{i_2,j_2}(y_0) \\
&\quad \cdot \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right)\left(\frac{\nu_1}{n_1} - x_0\right)^{p-k}\left(\frac{\nu_2}{n_2} - y_0\right)^k n_1^{i_1}(\nu_1 - n_1x_0)^{j_1} \\
&\quad \cdot n_2^{i_2}(\nu_2 - n_2y_0)^{j_2} p_{\nu_1,n_1}(x_0) p_{\nu_2,n_2}(y_0)
\end{aligned} \tag{16}$$

where the $h^*_{i_1,j_1}(x_0)$ are independent of ν_1 and n_1 and the $h^*_{i_2,j_2}(y_0)$ are independent of ν_2 and n_2. Hence it is sufficient to show that the inner double sum in (16), which we denote by σ, approaches zero under the restriction (3).

Now for any $\epsilon > 0$ there is a δ such that

$$|\alpha_k(x_1, y_1)| < \epsilon, \quad |x_1 - x_0| < \delta, \quad |y_1 - y_0| < \delta; \qquad k = 0, 1, 2, \ldots, p \tag{17}$$

(the choice of δ is made more precise later on), and there also exists an M such that $|\alpha_k(x_1, y_1)| \leq M$ for $0 \leq x_1 \leq 1$, $0 \leq y_1 \leq 1$.

In absolute value the part of σ in (16) corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| < \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| < \delta$$

does not exceed

$$n_1^{i_1-p+k} n_2^{i_2-k} \epsilon \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} |\nu_1 - n_1 x_0|^{p-k+j_1} |\nu_2 - n_2 y_0|^{k+j_2} p_{\nu_1,n_1}(x_0) p_{\nu_2,n_2}(y_0)$$

and by (5), this last expression does not exceed

$$C_2 \epsilon n_1^{i_1-p+k+\frac{1}{2}(p-k+j_1)} n_2^{i_2-k+\frac{1}{2}(k+j_2)}$$
$$\leqslant C_2 \epsilon (\sqrt{n_1})^{2i_1+j_1-(p-k)} (\sqrt{n_2})^{2i_2+j_2-k}.$$

As $2i_1 + j_1 \leqslant q$, $2i_2 + j_2 \leqslant p - q$, we have $2i_1 + 2i_2 + j_1 + j_2 - p \leqslant 0$, and so this last expression does not exceed $C_2\epsilon$ for all sufficiently large n_1, n_2 which satisfy (3).

In absolute value the part of σ in (16) corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| < \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| > \delta$$

does not exceed

$$n_1^{i_1-p+k} n_2^{i_2-k} M \sum_{\nu_1=0}^{n_1} |\nu_1 - n_1 x_0|^{p-k+j_1} p_{\nu_1,n_1}(x_0) \sum_{\left|\frac{\nu_2}{n_2} - y_0\right| > \delta} |\nu_2 - n_2 y_0|^{k+j_2} p_{\nu_2,n_2}(y_0).$$

Using (3) and the relations $|\nu - nu| \leqslant n$ and

$$\sum_{\nu_1=0}^{n_1} p_{\nu_1,n_1}(x_0) = 1,$$

we conclude that the above expression does not exceed

$$n_2^{i_1-p+k} n_2^{i_2-k} M_1 n_2^{p-k+j_1} n_2^{k+j_2} \sum_{\left|\frac{\nu_2}{n_2} - y_0\right| > \delta} p_{\nu_2,n_2}(y_0). \tag{18}$$

Provided n_1, n_2 are sufficiently large and $0 < \alpha < \frac{1}{2}$, the number $\delta = n_1^{-\alpha} + n_2^{-\alpha}$ satisfies the condition (17). Using the inequality (4) with this δ, we see that the expression (18) does not exceed

$$n_2^{i_1-p+k} n_2^{i_2-k} M_1 n_2^{p-k+j_1} n_2^{k+j_2} C_4 / n_2^{i_1+i_2+j_1+j_2+1},$$

which approaches zero as n_2 tends to infinity.

A similar conclusion holds for the part of σ corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| > \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| < \delta.$$

Finally, the part of σ corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| > \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| > \delta$$

is in absolute value not greater than

$$n_1^{j_1-p+k} n_2^{j_2-k} M \sum_{\left|\frac{\nu_1}{n_1}-x_0\right|>\delta} |\nu_1 - n_1 x_0|^{p-k+j_1} p_{\nu_1,n_1}(x_0) \sum_{\left|\frac{\nu_2}{n_2}-y_0\right|>\delta} |\nu_2 - n_2 y_0|^{k+j_2} p_{\nu_2,n_2}(y_0)$$

and by (4) this does not exceed

$$n_1^{j_1-p+k} n_2^{j_2-k} M C_5 \frac{n_1^{p-k+j_1}}{n_1^{j_1+j_2+1}} \frac{n_2^{k+j_2}}{n_2^{j_2+j_2+1}},$$

which approaches zero as $n_1, n_2 \to \infty$. Hence

$$\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B_{n_1,n_2}^{\beta_2}(x_0, y_0)$$

can be made arbitrarily small by taking n_1, n_2 large enough, satisfying (3). The theorem is now established.

We shall now show that the limit relation (2) also holds at the four corners (0, 0), (0, 1), (1, 0) and (1, 1) of the square S. It will be sufficient to discuss the corner (0, 0).

Writing

$$D_{\nu_1,\nu_2}(x, y) = \frac{d^q}{dx^q} x^{\nu_1} (1-x)^{n_1-\nu_1} \frac{d^{p-q}}{dy^{p-q}} y^{\nu_2} (1-y)^{n_2-\nu_2},$$

we have $D_{\nu_1,\nu_2}(0, 0) = 0$ for $\nu_1 \geqslant q+1$ or $\nu_2 \geqslant p-q+1$. For $\nu_1 \leqslant q$ and $\nu_2 \leqslant p-q$ by Leibniz's theorem, we have

$$\begin{aligned} D_{\nu_1,\nu_2}(0, 0) &= \binom{q}{\nu_1} \frac{d^{\nu_1}}{dx^{\nu_1}} x^{\nu_1} \frac{d^{q-\nu_1}}{dx^{q-\nu_1}} (1-x)^{n_1-\nu_1} \binom{p-q}{\nu_2} \frac{d^{\nu_2}}{dy^{\nu_2}} y^{\nu_2} \\ &\qquad \cdot \frac{d^{p-q-\nu_2}}{dy^{p-q-\nu_2}} (1-y)^{n_2-\nu_2} \Big|_{x=y=0} \\ &= (-1)^{q-\nu_1} \frac{q!}{(q-\nu_1)!} (n_1-\nu_1)(n_1-\nu_1-1) \dots (n_1-q+1) \\ &\qquad \cdot (-1)^{p-q-\nu_2} \frac{(p-q)!}{(p-q-\nu_2)!} (n_2-\nu_2) \dots (n_2-p+q+1) \\ &\simeq C_6 n_1^{q-\nu_1} n_2^{p-q-\nu_2}, \qquad n_1, n_2 \to \infty. \end{aligned} \tag{19}$$

Again we have to show that the limit relation (15) holds, but now at the point (0, 0) instead of (x_0, y_0).

We have

$$\begin{aligned} &\frac{\partial^p}{\partial x^q \partial y^{p-q}} B_{n_1,n_2}^{\beta_2}(x, y) \Big|_{x=y=0} \\ &\quad = \sum_{\nu_1=0}^{q} \sum_{\nu_2=0}^{p-q} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) \left(\frac{\nu_1}{n_1} - x\right)^{p-k} \left(\frac{\nu_2}{n_2} - y\right)^k \binom{n_1}{\nu_1} \binom{n_2}{\nu_2} D_{\nu_1,\nu_2}(x, y) \Big|_{x=y=0} \end{aligned}$$

and by (19), this in absolute value is

$$\simeq C_7 \left| \sum_{\nu_1=0}^{q} \sum_{\nu_2=0}^{p-q} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) \left(\frac{1}{n_1}\right)^{p-k} \left(\frac{1}{n_2}\right)^k n_1^{\nu_1} n_2^{\nu_2} n_1^{q-\nu_1} n_2^{p-q-\nu_2} \right|$$

and since n_1 and n_2 satisfy the inequality (3), the last expression does not exceed

$$C_3 \sum_{\nu_1=0}^{q} \sum_{\nu_2=0}^{p-q} \left| \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) \right|,$$

which approaches zero as $n_1, n_2 \to \infty$ because $\alpha_k(x, y) \to 0$ as $x \to 0$, $y \to 0$. Hence we have the relation

$$\frac{\partial^p}{\partial x^q \partial y^{p-q}} B^f_{n_1,n_2}(x, y)\Big|_{x=y=0} \to \frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x, y)\Big|_{x=y=0}$$

if the pth total differential of $f(x, y)$ exists at $(0, 0)$, provided n_1 and n_2 satisfy (3).

We now consider an example to show that the relation (2) does not necessarily hold at a boundary point of S different from one of the four corners. Consider the point $(x_0, 0)$, $0 < x_0 < 1$ and the function

$$f(x, y) = \epsilon(y)|x - x_0|, \qquad 0 \leqslant x \leqslant 1$$

where $\epsilon(y)$ is defined for $0 \leqslant y \leqslant 1$ and has the properties $\epsilon(y) \to 0$ and $y^{-1}\epsilon(y) \to +\infty$ as $y \to 0$. Take $n_1 = n_2 = n$. Obviously

$$\frac{d}{dy} p_{\nu,n}(y)\Big|_{y=0} = \begin{cases} n & \text{for } \nu = 1, \\ 0 & \text{for } \nu = 2, 3, \ldots, n. \end{cases}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial y} B_{nn}(x, y)\Big|_{(x_0,0)} &= \sum_{\nu_1=0}^{n} \left| \frac{\nu_1}{n} - x \right| p_{\nu_1,n}(x) \sum_{\nu_2=0}^{n} \epsilon\left(\frac{\nu_2}{n}\right) \frac{d}{dy} p_{\nu_2,n}(y)\Big|_{(x_0,0)} \\ &= \frac{1}{n} \sum_{\nu_1=0}^{n} \left| \nu_1 - nx_0 \right| p_{\nu_1,n}(x_0) \epsilon\left(\frac{1}{n}\right) n \\ &> C_9 \sqrt{n}\, \epsilon\left(\frac{1}{n}\right), \end{aligned}$$

which approaches $+\infty$ as $n \to \infty$ because it is known [2] that for every $0 < x < 1$ there exists a constant $C_9 > 0$ such that

$$\sum_{\nu=0}^{n} |\nu - nx|\, p_{\nu,n}(x) > C_9 \sqrt{n}, \qquad n \to \infty.$$

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University of Toronto

A CONVOLUTION TRANSFORM ADMITTING AN INVERSION FORMULA OF INTEGRO-DIFFERENTIAL TYPE

D. B. SUMNER

1. Introduction. Following Wiener's fundamental work on the Operational Calculus [**9**, pp. 557-584], Widder suggested [**5**, p. 219] that the inversion operator for the convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)\, dt \tag{1}$$

should be a suitable interpretation of $E(D)$, where $D = d/dx$ and the function $E(s)$ is defined by

$$\frac{1}{E(s)} = \int_{-\infty}^{\infty} e^{-st} G(t)\, dt. \tag{2}$$

Two methods of interpreting $E(D)$ have been used. The first, which appeals only to real-variable methods, has been used by Widder and his collaborators [**8**, p. 659; **5**, p. 217; **4**] in cases where $E(s)$ is entire with real zeros. The substance of the method is to express $E(s)$ as

$$\lim_{n\to\infty} E_n(s),$$

where $E_n(s)$ is a product of n factors of the form

$$\left(1 - \frac{s}{\lambda}\right)e^{s/\lambda},$$

and where, under suitable convergence conditions,

$$\lim_{n\to\infty} E_n(D) \cdot f(x)$$

can be computed.

The second method, which uses the complex variable, has been applied by Widder and Hirschmann [**8**, p. 692], and by the present author [**2**, p. 174] in cases where the entire function $E(s)$ can be expressed as a Fourier integral. It is naturally of less general applicability.

At the conclusion of their article Widder and Hirschmann conjectured the existence of convolution transforms, whose inversion operators are of integro-differential form, and not purely differential, as in all the then known cases. Widder has followed this by using such integro-differential operators to invert the Lambert transform [**6**, p. 171] and the Fourier sine transform [**7**, p. 119].

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In the last-named work, he showed that the methods of [8] were still applicable, even though the function defined by the integral

$$\frac{E(s)}{F(s)} = \int_{-\infty}^{\infty} e^{-st} G(t)\, dt \tag{2'}$$

was meromorphic. In both applications (to the Lambert and to the Fourier sine transform), a distinction was made between the factors $F(D)$ and $[E(D)]^{-1}$ of the inversion operator.

The objects of this note are:

(i) to show that transforms requiring an integro-differential invertor occur in quite simple cases, the example treated being one in which the kernel is a function little more complicated in form than the classical Stieltjes case [1, p. 473], where the kernel is $(1 + e^{-x})^{-1}$;

(ii) to prove that the order in which the factors of the invertor are applied is material.

We call $F(D)$ the differentiating factor, and $[E(D)]^{-1}$ the integrating factor. We use the methods of the complex variable, and interpret the factors of the invertor by expressing them as integrals.

2. Preliminary results. We consider the transform

$$f(x) = \int_{-\infty}^{\infty} H(x - t)\phi(t)\, dt, \tag{3}$$

where $H(x) = [e^{-2x} + 2e^{-x} \cos \pi\beta + 1]^{-1}$, and $0 < \beta < 1$. By adapting classical methods due to Widder [3, pp. 10-11], after an exponential change of variable, the following theorem is easily seen to be true:

THEOREM 1. *Let $f(x)$ be defined by (3), and $\phi(t)$ be such that the integral converges for at least one value of x in the strip $|\Im x| < \pi(\beta + 1)$, then it converges for all such x, and converges uniformly in any compact subset of the strip, so that $f(x)$ is analytic in the strip.*

It is well known that

$$\int_{-\infty}^{\infty} e^{-st} H(t)\, dt = \frac{\pi}{\sin \pi s} \cdot \frac{\sin \pi\beta(1 - s)}{\sin \pi\beta} \qquad (0 < \Re s < 2). \tag{4}$$

We take the reciprocal of this function, with $s = D$, as the invertor for equation (3), and we write

$$F(D) = \frac{\sin \pi D}{\pi} = \lim_{\theta \to \pi} \frac{1}{2\pi i} [e^{i\theta D} - e^{-i\theta D}] = \lim_{\theta \to \pi} F_\theta(D),$$

$$[E(D)]^{-1} = \frac{\sin \pi\beta}{\sin \pi\beta(1 - D)} = \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v} e^{vD}\, dv}{1 + e^{-v/\beta}}$$

and shall as usual interpret $e^{kD}(f)x$ as $f(x + k)$.

3. The inversion theorem. Our main result is the following theorem:

THEOREM 2. *Let $f(x)$ be defined by (3), with $0 < \beta < 1$, and $\phi(t)$ be such*

that the integral converges for at least one x in the strip $|\Im x| < \pi(\beta + 1)$. Then

$$F(D) \cdot [E(D)]^{-1} f(x) = \tfrac{1}{2}[\phi(x+) + \phi(x-)],$$

whenever the right-hand side has a meaning.

We have

$$\begin{aligned}[E(D)]^{-1} f(x) &= \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v} f(x+v)\, dv}{1 + e^{-v/\beta}}, \\ &= \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v}\, dv}{1 + e^{-v/\beta}} \int_{-\infty}^{\infty} H(x + v - t)\, \phi(t)\, dt, \\ &= \int_{-\infty}^{\infty} K(x - t)\, \phi\ (t)\, dt,\end{aligned}$$

where the interchange of the integrations is justified by the uniform convergence established in Theorem 1, and

$$K(r) = \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v} H(r+v)\, dv}{1 + e^{-v/\beta}}.$$

On considering the integral

$$\int \frac{1}{e^{-(r+w)} + 1} \frac{e^{-w}\, dw}{1 - e^{-w/\beta}}$$

taken round the rectangle with vertices $R \pm \pi i\beta$, $-R \pm \pi i\beta$, and making $R \to \infty$, we see that

$$K(r) = [1 + e^{-r}]^{-1}.$$

Thus

$$[E(D)]^{-1} f(x) = \int_{-\infty}^{\infty} \frac{\phi(t)\, dt}{1 + e^{-(x-t)}}. \tag{5}$$

This is the classical Stieltjes transform (after an exponential change of variable); and if we denote it by $g(x)$, it is well known [**4**, p. 340] that

$$\lim_{\theta \to x} F_\theta(D) \cdot g(x) = \tfrac{1}{2}[\phi(x+) + \phi(x-)].$$

Theorem 2 is thus seen to be true.

We do not here allow the value $\beta = 0$, since the integral (4) then has the value $\pi(1 - s)/\sin \pi s$, whose reciprocal has no poles. It is easily seen that in this case, after suitable changes of variable, the equation (3) takes the form

$$h(x) = \int_0^{\infty} \frac{\psi(t)\, dt}{(x + t)^{\frac{1}{2}}}$$

which has been discussed by the present author [**2**].

4. Commutativity of the factors in the operator. That the order in which the factors $F(D)$ and $[E(D)]^{-1}$ were applied to the function defined by (3) is material, is brought out by the following example.

Let

$$\phi(t) = \begin{cases} e^t, & t > 0, \\ 0, & t < 0. \end{cases}$$

It is then easily seen that the corresponding

$$f(x) = \int_0^\infty H(x-t)\cdot e^t\,dt,$$
$$= \frac{e^x}{\sin \pi\beta}\cdot \arctan\left[\frac{\sin \pi\beta}{e^{-x}+\cos \pi\beta}\right];$$

while

$$F_\theta(D)f(x) = \frac{1}{\sin \pi\beta}\left[e^{x+i\theta}\arctan\left(\frac{\sin \pi\beta}{e^{-x-i\theta}+\cos \pi\beta}\right)\right.$$
$$\left. - e^{x-i\theta}\arctan\left(\frac{\sin \pi\beta}{e^{-x+i\theta}+\cos \pi\beta}\right)\right] \to 0$$

as $\theta \to \pi$. The differentiating factor, if applied first, is thus seen to annihilate the function, a fact which makes pointless the subsequent application of the integrating factor.

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Hamilton College, McMaster University

SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WHICH GENERALIZE THE HEAT EQUATION

I. I. HIRSCHMAN, JR.

1. Introduction. The distribution of heat in an infinite rod is closely bound up with the theory of the Weierstrass transform. This connection is exhibited most clearly in the theory of Widder [3]. Consider the heat equation

$$\frac{\partial u}{\partial h} = \frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

Widder has shown that if $u(x, h)$ satisfies 1.1 and if $u(x, h) \geqslant 0$ for $-\infty < x < \infty$, $a < h < b$ then we have

$$u(x, h'') = \int_{-\infty}^{\infty} k(x - y, h'' - h')u(y, h')\, dy \tag{1.2}$$

if $a < h'$, $h'' < b$, $h'' > h'$. Here

$$k(x, t) = (4\pi t)^{-\frac{1}{2}} \exp\,(-x^2/4t) \tag{1.3}$$

$$k(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp\,[-ty^2 - ixy]\, dy. \tag{1.4}$$

Let $\mathfrak{E}_m$ and $\mathfrak{H}_n$ be real m- and n-dimensional vector spaces respectively. We shall write the elements X, Y, Z of $\mathfrak{E}_m$ and H, H', H'' of $\mathfrak{H}_n$ as column matrices,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_m \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_m \end{pmatrix},$$

$$H = \begin{pmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ h_n \end{pmatrix}, \quad H' = \begin{pmatrix} h_1' \\ h_2' \\ \cdot \\ \cdot \\ h_n' \end{pmatrix}, \quad H'' = \begin{pmatrix} h_1'' \\ h_2'' \\ \cdot \\ \cdot \\ h_n'' \end{pmatrix}.$$

In place of $u(x, h)$, we consider $u(X, H)$ depending upon the $m + n$ real variables $x_1, \ldots, x_m$, $h_1, \ldots, h_n$, and instead of the heat equation 1.1, we consider the system of partial differential equations

$$\frac{\partial u}{\partial h_l} = \sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial x_i\, \partial x_j} a_{ij}^l \qquad l = 1, \ldots, n. \tag{1.5}$$

Here the a's are real constants and

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$$a_{ij}^{l} = a_{ji}^{l}.$$

In the present paper we shall develop for such systems a theory analogous to that of Widder.

Let $\mathfrak{T}_{\frac{1}{2}(m+1)}$ be the space of real symmetric $m \times m$ matrices,

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ t_{21} & t_{22} & \cdots & t_{2m} \\ \cdots & & \cdots & \\ t_{m1} & t_{m2} & \cdots & t_{mm} \end{pmatrix}, \qquad T' = \begin{pmatrix} t'_{11} & t'_{12} & \cdots & t'_{1m} \\ t'_{21} & t'_{22} & \cdots & t'_{2m} \\ \cdots & & \cdots & \\ t'_{m1} & t'_{m2} & \cdots & t'_{mm} \end{pmatrix}.$$

We may define a partial ordering in $\mathfrak{T}_{\frac{1}{2}m(m+1)}$ by writing $T' > T$ if $T' - T$ is positive definite. We introduce corresponding to the system 1.5 a mapping $H \to H_a$ of $\mathfrak{H}_n$ into $\mathfrak{T}_{\frac{1}{2}m(m+1)}$, the (i, j) entry of H_a being defined as

$$\sum_{l=1}^{n} a_{ij}^{l} h_l \qquad (i, j = 1, 2, \ldots, m).$$

We shall show that if $u(X, H)$ satisfies 1.5 and if $u(X, H) \geqslant 0$ for $X \in \mathfrak{E}_m$, $H \in \mathfrak{R}$ where $\mathfrak{R}$ is an open convex subset of $\mathfrak{H}_n$, then

$$1.6 \qquad u(X, H'') = \int_{\mathfrak{E}_m} k(X - Y, H''_a - H'_a) u(Y, H')\, dY$$

if H', $H'' \in \mathfrak{R}$, $H''_a > H'_a$. The function $k(X, T)$ is defined by the formula

$$1.7 \qquad k(X, T) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp\left[- Y^*TY - iY^*X \right] dY;$$

here T must be positive definite. An explicit formula for $k(X, T)$ (due to Czüber) is given in §3.

If A is any matrix, A^* is its transpose; thus

$$Y^*X = \sum_{i=1}^{m} y_i x_i,$$

$$Y^*TY = \sum_{i,j=1}^{m} y_i t_{ij} y_j.$$

2. A special case. The equation of heat transfer in m-dimensional space is

$$2.1 \qquad \frac{\partial u}{\partial h} = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} \quad \text{or} \quad \frac{\partial u}{\partial h} = \Delta u.$$

In this section we shall establish our theory for this special case. If $m = 1$ in 2.1 we have the equation considered by Widder. It is to be noted that only relatively minor adjustments are needed to make the extension from 1 to m dimensions. We set

$$2.2 \qquad k(X, t) = (4\pi t)^{-\frac{1}{2}m} \exp\left[- \frac{1}{4t} X^*X \right],$$

$$k(X, t) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp\left[- tY^*Y - iY^*X \right] dY.$$

Let us write $|X|$ for

$$(X^*X)^{\frac{1}{2}} = (\sum_1^m x_i^2)^{\frac{1}{2}}.$$

We begin with a discussion of the convolution transform

$$\psi(X, h) = \int_{\mathfrak{C}_m} k(X - Y, h)\phi(Y)\, dY \tag{2.3}$$

under radial symmetry. A function ϕ defined on $\mathfrak{C}_m$ is said to be radically symmetric if $\phi(X') = \phi(X'')$ whenever $|X'| = |X''|$. If in formula 2.3 $\phi(Y)$ is radially symmetric, then for each h, $\psi(X, h)$ is also radially symmetric and the m-fold integration in our formula may be collapsed to a single integration with respect to a new kernel, which we need later and which we now compute. Since $\phi(Y)$ depends only upon $|Y| = \rho$ we write $\phi(\rho)$ instead of $\phi(Y)$. Similarly since $\psi(X, h)$ depends only upon $|X| = r$ and h, we write $\psi(r, h)$ instead of $\psi(X, h)$.

Introducing spherical coordinates,

$$\begin{aligned}
y_1 &= \rho \cos \phi_1 \\
y_2 &= \rho \sin \phi_1 \cos \phi_2 \\
y_3 &= \rho \sin \phi_2 \sin \phi_2 \cos \phi_3 \\
&\cdots \\
y_{m-1} &= \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{m-2} \cos \phi_{m-1} \\
y_m &= \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{m-2} \sin \phi_{m-1},
\end{aligned}$$

and setting $x_1 = r$, $x_2 = x_3 = \ldots = x_m = 0$, we have

$$\psi(r, h) = (4\pi t)^{-\frac{1}{2}m} \int_0^\infty \int_{-\pi}^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \phi(\rho) e^{-(r^2-2r\rho\cos\phi_1+\rho^2)/4t}$$
$$\rho^{m-1} \sin^{m-2}\phi_1 \ldots \sin\phi_{m-2}\, d\phi_1 \ldots d\phi_{m-1}\, d\rho.$$

From this we obtain

$$\psi(r, t) = (4\pi t)^{-\frac{1}{2}m} \frac{2\pi^{\frac{1}{2}(m-3)}}{\Gamma(\frac{1}{2}m - \frac{1}{2})} \int_0^\infty \phi(\rho) e^{-(r^2+\rho^2)/4t} \rho^{m-1} d\rho \int_0^{\pi} e^{r\rho\cos\phi_1/2t} \sin^{m-2}\phi_1\, d\phi_1.$$

By Watson [2, p. 79] we have

$$\int_0^{\pi} e^{r\rho\cos\phi_1/2t} \sin^{m-2}\phi_1\, d\phi_1 = \frac{\Gamma(\frac{1}{2}m - \frac{1}{2})\sqrt{\pi}}{(r\rho/4t)^{\frac{1}{2}m-1}} I_{\frac{1}{2}m-1}\left(\frac{r\rho}{2t}\right).$$

We finally get

$$\psi(r, t) = (2t)^{-1} \int_0^\infty e^{-(r^2+\rho^2)/4t} \rho^{\frac{1}{2}m} r^{-\frac{1}{2}m+1} I_{\frac{1}{2}m-1}\left(\frac{r\rho}{2t}\right) \phi(\rho)\, d\rho. \tag{2.4}$$

The function $u(X, h)$ is said to belong to S, $u(X, h) \in S$, at (X_0, h_0) if there exists an open set in $E_m \times H$ containing (X_0, h_0), throughout which the partial derivatives

$$\frac{\partial u}{\partial h}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (i, j = 1, \ldots, m)$$

are continuous and if

$$\frac{\partial u}{\partial h} = \Delta u \qquad \text{for } X = X_0, h = h_0.$$

We denote the set of points

$$|X| < R, \qquad 0 < h \leqslant c$$

by D_{Rc} and the set of points

$$|X| \leqslant R, \qquad t = 0,$$

$$|X| = R, \qquad 0 < t \leqslant c,$$

by B_{Rc}.

LEMMA 2a. *If*

1. $u(X, h) \in S$ *for* $(X, h) \in D_{Rc}$;
2. $\lim_{X \to X_0, h \to h_0} u(X, h) \geqslant 0$ *for* $(X_0, h_0) \in B_{Rc}$

then $u(X, h) \geqslant 0$ *in* D_{Rc}.

Suppose that there exists a point $(X_1, h_1) \in D_{Rc}$ at which we have $u(X_1, h_1) = -l < 0$. Form the function

$$v(X, h) = u(X, h) + k(h - h_1)$$

where $k > 0$ is chosen so small that

$$\lim_{X \to X_0, h \to h_0} v(X, h) \geqslant -\tfrac{1}{2}l \qquad (X_0, h_0) \in B_{Rc}.$$

For $\delta > 0$ suitably chosen $v(X, h) > -l$ within a distance δ of B_{Rc}. Consequently the minimum of $v(X, h)$ in D_{Rc} is attained at some point (X_2, h_2) of D_{Rc}. The function $v(X, h)$ satisfies the partial differential equation

2.5 $$\Delta v = \frac{\partial v}{\partial h} - k.$$

At (X_2, h_2) we must have

2.6 $$\Delta v \geqslant 0,$$

2.61 $$\frac{\partial v}{\partial h} \leqslant 0.$$

(If $h_2 \neq c$ the equality holds in 2.61.) Equations 2.5 and 2.6, 2.61 are in contradiction.

LEMMA 2b. *If*

1. $u(X, h) \in S,$ $X \in \mathfrak{E}_m, 0 < h \leqslant c;$
2. $\lim_{X \to X_0, h \to 0+} u(X, h) = 0,$ $X_0 \in \mathfrak{E}_m;$
3. $M(r) = \text{l.u.b.}_{|X| = r,\ 0 < h < c} |u(X, h)|;$
4. $M(r) = O(e^{ar^2}),$ $r \to +\infty$

for some $a > 0$; *then* $u(X, H) \equiv 0$ *for* $X \in \mathfrak{E}_m$, $0 < h \leqslant c$.

Take $\lambda > 1$ and form the auxiliary function

$$U_R(X, h) = \lambda M(R)(\pi c)^{\frac{1}{2}} h^{-1} e^{-(|X|^2+R^2)/4h} R^{\frac{1}{2}m} |X|^{-\frac{1}{2}m+1} I_{\frac{1}{2}m-1}\left(\frac{R|X|}{2h}\right).$$

Using the asymptotic formula

$$I_\nu \sim e^x / \sqrt{2\pi x} \qquad (x \to +\infty), \tag{2.7}$$

we find that [2]

$$U_R(X, h)|_{|X|=R} \backsim \lambda(c/h)^{\frac{1}{2}} M(R), \qquad R \to \infty,$$

uniformly for $0 < h \leqslant c$. Thus when R is large

$$U_R(X, h) > M(R), \qquad |X| = R, 0 < h \leqslant c.$$

The function $U_R(X, h) - u(X, h)$ belongs to S for $X \in \mathfrak{E}_m$, $0 < h \leqslant c$. When R is large

$$\varliminf_{X \to X_0,\ h \to h_0} U_R(X, h) - u(X, h) \geqslant 0, \qquad (X_0, h_0) \in B_{Rc}.$$

By Lemma 2a we have

$$u(X, h) \leqslant U_R(X, h), \qquad (X, h) \in D_{Rc}.$$

Fix X and h and let $R \to +\infty$. Making use of 2.7 we find that $u(X, h) = 0$ for $0 < h < 1/4a$, $X \in \mathfrak{E}_m$. If $4a \leqslant 1/c$, our proof is complete. Otherwise repeat the above argument with $u(X, h)$ replaced by $u(X, h + 1/4a)$, and so forth.

$L(\mathfrak{E}_m)$ is the class of functions $\phi(X)$ defined for $X \in \mathfrak{E}_m$ and such that

$$\int_{\mathfrak{E}_m} |\phi(X)|\, dX$$

exists.

Exactly as in [3] we may establish

LEMMA 2c. *If*

1. $\phi(X)e^{-a|X|^2} \in L(\mathfrak{E}_m)$ *for some* $a > 0$,

2. $F(X, h) = \int_{\mathfrak{E}_m} k(X - Y, h)\phi(Y)\, dY$,

then $F(X, h)$ *is defined and belongs to* S *in the strip* $0 < h < 1/4a$.

LEMMA 2d. *Under the assumptions of Lemma* 2c, *we have*

$$\varliminf_{X \to X_0} \phi(X) \leqslant \varliminf_{X \to X_0,\ h \to 0+} F(X, h); \quad \varlimsup_{X \to X_0,\ h \to 0+} F(X, h) \leqslant \varlimsup_{X \to X_0} \phi(X).$$

LEMMA 2e. *If* $\phi(Y)$ *is integrable for* $|Y| \leqslant A$ *then for any* $c > 0$, *we have*

$$\lim_{|X| \to \infty} \left(\text{l.u.b.}_{0<h<c} \left| \int_{|Y| \leqslant A} k(X - Y, h)\phi(Y)\, dY \right|\right) = 0.$$

LEMMA 2f. *If*

1. $u(X, h) \in S$, $\qquad 0 < h < c, X \in \mathfrak{E}_m$;
2. $u(X, h) \geqslant 0$, $\qquad 0 < h < c, X \in \mathfrak{E}_m$;

then we have

$$\int_{\mathfrak{E}_m} k(X - Y, \delta')u(Y, \delta)\,dY \leqslant u(X, \delta' + \delta),$$
$$0 < \delta, 0 < \delta', \delta + \delta' < c, X \in \mathfrak{E}_m.$$

Consider the function

$$v_A(X, h) = u(X, h + \delta) - \int_{|Y|\leqslant A} k(X - Y, h)u(Y, \delta)\,dY$$

where A is a positive constant. If $\delta' < c' < c - \delta$ then $v_A(X, h)$ belongs to S for $(X, h) \in D_{Rc'}$ for any R. Let $l > 0$ be given. Using Lemmas 2d and 2e we see that, if R is sufficiently large,

$$\varliminf_{X\to X_0, h\to h_0} v_A(X, h) \geqslant -l, \qquad (X_0, h_0) \in B_{R,c'}.$$

Lemma 2a implies that $v_A(X, h) \geqslant -l$ for $(X, h) \in D_{R,c'}$. Letting $l \to 0+$ and $R \to +\infty$, we find that $v_A(X, h) \geqslant 0$ for $X \in \mathfrak{E}_m$, $0 < h < c'$. Setting $h = \delta'$ we have

$$\int_{|Y|\leqslant A} k(X - Y, \delta')u(Y, \delta)\,dY \leqslant u(X, \delta' + \delta).$$

Letting A increase without limit, we obtain our desired result.

LEMMA 2g. *If*

1. $u(X, h) \in S$, $0 \leqslant h < c$, $X \in \mathfrak{E}_m$;
2. $u(X, h) \geqslant 0$, $0 \leqslant h < c$, $X \in \mathfrak{E}_m$;
3. $u(X, 0) = 0$, $X \in \mathfrak{E}_m$;

then $u(X, h) = 0$ $0 \leqslant h < c$, $X \in \mathfrak{E}_m$.

We set

$$w(X, h) = \int_0^h u(X, t)\,dt.$$

It is easily verified that $w(X, h) \in S$ for $0 \leqslant h < c$, $X \in \mathfrak{E}_m$. We have $w(X, h_1) \geqslant w(X, h_2)$ if $c > h_1 \geqslant h_2 \geqslant 0$. It follows that

$$\frac{\partial}{\partial h} w(X, h) \geqslant 0$$

and hence that $\Delta w(X, h) \geqslant 0$ for $0 \leqslant h < c$; equivalently $w(X, h)$ is a subharmonic function of X for each value of h, $0 \leqslant h < c$.

Let δ be an arbitrary number $0 < \delta < c$ and let δ' be such that $0 < \delta' < c - \delta$. By Lemma 2f we have

$$w(0, \delta' + \delta) \geqslant (4\pi\delta)^{-\frac{1}{2}m} \int_{\mathfrak{E}_m} e^{-|Y|^2/4\delta} w(Y, \delta')\,dY.$$

Again we have

$$\int_{\mathfrak{E}_m} e^{-|Y|^2/4\delta} w(Y, \delta')\,dY \geqslant M(X) \int_{|Y-X|\leqslant 1} w(Y, \delta')\,dY$$

where

$$M(X) = \text{g.l.b.}_{|Y-X|<1} [e^{-|Y|^2/4\delta}].$$

Since $w(Y, \delta')$ is subharmonic

$$\int_{|Y-X|<1} w(Y, \delta')\, dY \geqslant \pi w(X, \delta').$$

It follows that

$$w(X, \delta') = O(\exp a|X|^2)$$

for $a > 1/4\delta$. Since

$$\frac{\partial}{\partial h} w\,(X, h) \geqslant 0$$

we find that

$$\text{l.u.b.}_{0<\delta<\delta', |X|=R} |w(X, \delta)| = O(e^{aR^2}).$$

Applying Lemma 2b we see that $w(X, h) = 0$ for $0 \leqslant h \leqslant \delta'$, $X \in \mathfrak{E}_m$. Since $\delta' < c$ is arbitrary we have $w(X, h) = 0$ for $0 \leqslant h < c$, $X \in \mathfrak{E}_m$. This in turn implies that $u(X, h) = 0$ for $X \in \mathfrak{E}_m$, $0 \leqslant h < c$.

THEOREM 2h. *If*

1. $u(X, h) \in S$ *for* $X \in \mathfrak{E}_m$, $a < h < b$;
2. $u(X, h) \geqslant 0$ *for* $X \in \mathfrak{E}_m$, $a < h < b$;
3. $a < h'$, $h'' < b$, $h'' > h'$;

then

$$u(X, h'') = \int_{\mathfrak{E}_m} k(X - Y, h'' - h')u(Y, h')\, dY.$$

By Lemma 2f we have

$$\int_{\mathfrak{E}_m} k(X - Y, h - h')u(Y, h') \leqslant u(X, h)$$

for $h' < h < b$. This together with Lemma 2c implies that

$$\int_{\mathfrak{E}_m} k(X - Y, h - h')u(Y, h')\, dY \in S$$

for $h' < h < b$, $X \in \mathfrak{E}_m$. Thus

$$v(X, h) = u(X, h) - \int_{\mathfrak{E}_m} k(X - Y, h - h')u(Y, h')\, dY$$

belongs to S and is non-negative for $h' < h < b$, $X \in \mathfrak{E}_m$. Moreover, by Lemma 2d,

$$\lim_{h \to h',\ Y \to X_0} v(X, h) = 0.$$

Lemma 2g implies that $v(X, h) \equiv 0$. Setting $h = h''$, we obtain the desired result.

3. The main theorem. Let $T \in \mathfrak{T}_{\frac{1}{2}m(m+1)}$ be positive definite and let $X \in \mathfrak{E}_m$. We set

3.1 $$k(X, T) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp[-Y^*TY - iY^*X]\, dY.$$

Since T is positive definite, there exists a constant $\epsilon > 0$ such that $Y^*TY \geq \epsilon |Y|^2$. This insures the validity of our definition. Let $\Phi = [\phi_{ij}]_{i,j=1,\ldots,m}$ be a real unimodular matrix. We assert that

3.2 $$k(\Phi^{-1}X, T) = k(X, \Phi T\Phi^*).$$

We have

$$k(\Phi^{-1}X, T) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp[-Y^*TY - iY^*\Phi^{-1}X]\, dY.$$

Making the change of variable $Y = \Phi^*Z$ we obtain

$$k(\Phi^{-1}X, T) = (2\pi)^{-m} \int_{\mathfrak{E}_m} \exp[-Z^*\Phi T\Phi^*Z - iZX]\, dZ = k(X, \Phi T\Phi^*).$$

This formula may be used to compute $k(X, T)$ explicitly, see [1, p. 185]. By this method, Czüber has shown that

3.3 $$k(X, T) = \frac{\exp\left(\begin{vmatrix} 0 & X^* \\ X & T \end{vmatrix} \Big/ 4|T|\right)}{(4\pi)^{\frac{1}{2}m}|T|^{\frac{1}{2}}}.$$

In particular if T is a diagonal matrix with equal entries,

$$T = \begin{pmatrix} \tau & 0 & \ldots & 0 \\ 0 & \tau & \ldots & 0 \\ \ldots & \ldots & & \\ 0 & 0 & \ldots & \tau \end{pmatrix},$$

then $k(X, T)$ is equal to $k(X, \tau)$.

Consider the system of partial differential equations

3.4 $$\frac{\partial u}{\partial h_l} = \sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial x_i\, \partial x_j} a_{ij}^l \qquad (a_{ij}^l = a_{ji}^l;\ l = 1, \ldots, n),$$

where the a's are real constants. The function $u(X, H)$ is said to belong to $S(a)$ at (X_0, H_0) if there exists an open set in $E_m \times H_n$ containing (X_0, H_0) throughout which the partial derivatives

$$\frac{\partial u}{\partial h_l}\,(l = 1, \ldots, n), \qquad \frac{\partial^2 u}{\partial x_i\, \partial x_j}\,(i, j = 1, \ldots, m)$$

are continuous and if equation 3.4 holds for $X = X_0$, $H = H_0$.

LEMMA 3a. *If*:

1. $u(X, H) \in S(a)$, *where* $\mathfrak{N}$ *is an open subset of* $\mathfrak{H}_n$, $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$;
2. $\Phi = [\phi_{ij}]_{i,j=1,\ldots,m}$ *is a real unimodular matrix;*
3. $w(X, h) = u(\Phi^{-1}X, h)$;

then

3.5 $$\frac{\partial w}{\partial h_l} = \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} b^l_{\alpha,\beta}$$

for $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$, *where*

$$b^l_{\alpha,\beta} = \sum_{i,j=1}^{m} a^l_{ij}\, \phi_{\alpha i}\, \phi_{\beta j}.$$

(Briefly we have $w(X, H) \in S(b)$ for $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$.)

We have $w(X, H) = u(Y, H)$ where $X = \Phi Y$. Now

$$\frac{\partial u}{\partial h_l} = \frac{\partial w}{\partial h_l}, \quad \frac{\partial^2 u}{\partial y_i\, \partial y_j} = \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta}\, \phi_{\alpha i}\, \phi_{\beta j}.$$

Substituting in 3.4 we obtain 3.5.

If $H \in \mathfrak{H}_n$, then $H \to H_b$ is a mapping of $\mathfrak{H}_n$ into $\mathfrak{T}_{\frac{1}{2}m(m+1)}$, the (i, j) entry of H_b being

$$\sum_{l=1}^{n} h_l\, b^l_{i,j}.$$

It is easily verified that

3.6 $$\Phi H_a\, \Phi^* = H_b.$$

LEMMA 3b. *If*:

1. $u(X, H)$ *is continuous for* $X \in \mathfrak{E}_m$, $H \in \mathfrak{N}$ *where* $\mathfrak{N}$ *is an open subset of* $\mathfrak{H}_n$,
2. $H', H'' \in \mathfrak{N}$, $H''_a > H'_a$,
3. $u(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_a - H'_a) u(Y, H')\, dY$,
4. $w(X, H) = u(\Phi^{-1}X, H)$,

then $w(X, H)$ *is defined and continuous for* $X \in E_m$, $H \in N$, $H''_b - H'_b > 0$, *and*

$$w(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_b - H'_b) w(Y, H_r)\, dY.$$

In 3, replace X by $\Phi^{-1}X$ and make the change of variable $Y = \Phi^{-1}Z$ to obtain

$$u(\Phi^{-1}X, H'') = \int_{\mathfrak{E}_m} k(\Phi^{-1}Z - \Phi^{-1}X, H''_a - H'_a) u(\Phi^{-1}Z, H')\, dZ.$$

$$w(X, H'') = \int_{\mathfrak{E}_m} k(\Phi^{-1}Z - \Phi^{-1}X, H''_a - H'_a) w(Z, H')\, dZ.$$

By 3.2 and 3.6 we have

$$k(\Phi^{-1}Z - \Phi^{-1}X, H''_a - H'_a) = k(Z - X, \Phi H''_a\, \Phi^* - \Phi H'_a\, \Phi^*),$$
$$= k(Z - X, H''_b - H'_b).$$

Our lemma follows.

THEOREM 3c. *If*:

1. $u(X, H) \in S(a)$, $X \in E_m$, $H \in \mathfrak{R}$, *where* $\mathfrak{R}$ *is an open convex subset of* $\mathfrak{H}_n$,
2. $u(X, H) \geqslant 0$, $\quad X \in \mathfrak{E}_m, H \in \mathfrak{R}$
3. $H', H'' \in \mathfrak{R}$, $\quad H''_a > H'_a$,

then

$$u(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_a - H'_a)u(Y, H')\, dY.$$

Let Φ be a real unimodular $m \times m$ matrix. Because of Lemmas 3a and 3b, it is sufficient to establish the corresponding relation for $w(X, H) = u(\Phi^{-1}X, H)$,

$$w(X, H'') = \int_{\mathfrak{E}_m} k(Y - X, H''_b - H'_b)w(Y, H')\, dY.$$

By 3.6 we have

$$H''_b - H'_b = \Phi(H''_a - H_a)\Phi^*.$$

It is possible to choose Φ so that $H''_b - H'_b$ is a diagonal matrix with equal entries, i.e.,

$$H''_b - H'_b = \begin{pmatrix} \tau & 0 & \dots & 0 \\ 0 & \tau & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \tau \end{pmatrix}.$$

Having chosen Φ in this manner, we consider

$$w\left[X, \left(1 - \frac{t}{\tau}\right)H' + \frac{t}{\tau}H''\right] = v\,(X, t).$$

Since $\mathfrak{R}$ is convex and open, $v(X, t)$ is defined for $a < t < b$ where $a < 0$, $b > \tau$. We have

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{n} \frac{\partial w}{\partial h_i}\left[\frac{1}{\tau}(h''_i - h'_i)\right].$$

Since

$$\frac{\partial w}{\partial h_i} = \sum_{i,j=1}^{m} \frac{\partial^2 w}{\partial x_i\, \partial x_j}\, b^i_{ij} = \sum_{i,j=1}^{m} \frac{\partial^2 v}{\partial x_i\, \partial x_j}\, b^i_{ij}$$

we find that

$$\frac{\partial v}{\partial t} = \frac{1}{\tau} \sum_{i,j=1}^{m}\left[\sum_{i=1}^{n} b^i_{ij}\left(h''_i - h'_i\right)\right]\frac{\partial^2 v}{\partial x_i\, \partial x_j},$$

$$\frac{\partial v}{\partial t} = \Delta v.$$

Thus $v(X, t) \in S$ for $X \in \mathfrak{E}_m$, $a < t < b$ where $a < 0$, $b > \tau$. Applying Theorem 2h we have

$$v(X, \tau) = \int_{\mathfrak{E}_m} k(X - Y, \tau)v(Y, 0)\, dY.$$

Now

$$v(X, \tau) = w(X, H''), \quad v(X, O) = w(X, H').$$

Thus

$$w(X, H'') = \int_{\mathfrak{E}_m} k(X - Y, \tau) w(Y, H')\, dY,$$

$$w(X, H'') = \int_{\mathfrak{E}_m} k(X - Y, H''_b - H'_b) w(Y, H')\, dY,$$

a relation which we have seen to be quivalent to our theorem.

There are some systems of equations 3.4 for which Theorem 3c gives no information. This is because there do not exist real vectors H' and H'' such that $H''_a > H'_a$. The system consisting of the single equation

$$\frac{\partial u}{\partial h} = \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}$$

is of this type.

Making use of Theorem 3c and the concept of weak compactness, we may demonstrate the following result.

THEOREM 3d. *Let $\mathfrak{R}$ be an open subset of $\mathfrak{H}_n$ such that $H', H'' \in \mathfrak{R}$ imply $\lambda''H'' + \lambda'H' \in \mathfrak{R}$ for $0 < \lambda'$, $0 < \lambda''$, $\lambda' + \lambda'' \leqslant 1$, and such that $H \in \mathfrak{R}$ implies that H_a is positive definite. If:*

1. $u(X, H) \in S\,(a)$, $\qquad X \in \mathfrak{E}_m, H \in \mathfrak{R}$
2. $u(X, H) \geqslant 0$, $\qquad X \in \mathfrak{E}_m, H \in \mathfrak{R}$

then

$$u(X, H) = \int_{\mathfrak{E}_m} k(X - Y, H_a)\, dm(Y) \qquad X \in \mathfrak{E}_m, H \in \mathfrak{R},$$

where $m(Y)$ is a non-negative measure defined in the σ-field of Borel sets of E_m.

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Washington University,
Saint Louis